In Honour of Torben Krarup

Collocation with Integer Parameter Trend

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Linear model for prediction

$$\begin{matrix}\rightarrow \\ \leftarrow\end{matrix}\begin{bmatrix} \underline{y} \\ \underline{y}_0 \end{bmatrix} = \begin{bmatrix} A \\ A_0 \end{bmatrix} x + \begin{bmatrix} \underline{e} \\ \underline{e}_0 \end{bmatrix}$$

$\underline{y}$ & $\underline{y}_0$	:	observable and unobservable random vector
$\underline{e}$ & $\underline{e}_0$	:	zero-mean random vector with known dispersion
A & A_0	:	known matrices
x	:	unknown integer parameter vector

Random parameter vector with unknown integer mean x : $\underline{y} = A\underline{x} + \underline{n}$

$$\begin{matrix}\rightarrow \\ \leftarrow\end{matrix}\begin{bmatrix} \underline{y} \\ \underline{x} \end{bmatrix} = \begin{bmatrix} A \\ I \end{bmatrix} x + \begin{bmatrix} A(\underline{x} - x) + \underline{n} \\ \underline{x} - x \end{bmatrix}$$

Collocation with integer trend: $\underline{y} = Ax + \underline{s} + \underline{n}, \quad \underline{s}_0$

$$\begin{matrix}\rightarrow \\ \leftarrow\end{matrix}\begin{bmatrix} \underline{y} \\ \underline{s}_0 \end{bmatrix} = \begin{bmatrix} A \\ 0 \end{bmatrix} x + \begin{bmatrix} \underline{s} + \underline{n} \\ \underline{s}_0 \end{bmatrix}$$

Kalman filter with integer state:

$$\begin{matrix}\rightarrow \\ \leftarrow\end{matrix}\begin{bmatrix} \hat{\underline{x}}_{k|k-1} \\ \underline{y}_k \end{bmatrix} = \begin{bmatrix} I \\ A_k \end{bmatrix} x_k + \begin{bmatrix} \underline{d}_{k|k-1} \\ \underline{e}_k \end{bmatrix}$$

Integer-based least-squares prediction

$$\begin{bmatrix} \underline{y} \\ \underline{y}_0 \end{bmatrix} = \begin{bmatrix} A \\ A_0 \end{bmatrix} x + \begin{bmatrix} \underline{e} \\ \underline{e}_0 \end{bmatrix} ; x \in \mathbb{Z}^n ; \quad \begin{bmatrix} Q_{yy} & Q_{yy_0} \\ Q_{y_0 y} & Q_{y_0 y_0} \end{bmatrix}$$

Predictor: $\underline{\check{y}}_0 = A_0 \underline{\check{x}} + Q_{y_0 y} Q_{yy}^{-1} (\underline{y} - A \underline{\check{x}})$

with
$$\begin{aligned} \underline{\check{x}} &= \arg\min_{z \in \mathbb{Z}^n} \| \underline{\hat{x}} - z \|^2_{Q_{\hat{x}\hat{x}}} \\ \underline{\hat{x}} &= (A^T Q_{yy}^{-1} A)^{-1} A^T Q_{yy}^{-1} \underline{y} \end{aligned}$$

Prediction error: $\underline{\check{e}}_0 = \underline{y}_0 - \underline{\check{y}}_0$

If $x \in \mathbb{R}^n$, then:

- mean: $E(\underline{\check{e}}_0) = 0$
- error-variance: $Q_{\check{e}_0 \check{e}_0} = Q_{y_0 y_0|y} + A_{0|y} Q_{\check{x}\check{x}} A_{0|y}^T$
- distribution: $\underline{\check{e}}_0 \sim \mathrm{N}(0, Q_{\check{e}_0 \check{e}_0})$

If $x \in \mathbb{Z}^n$, then $\underline{\check{e}}_0$ <u>not</u> normally distributed

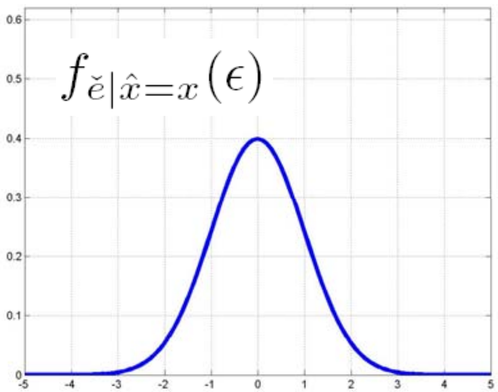
Distribution (PDF) of prediction error

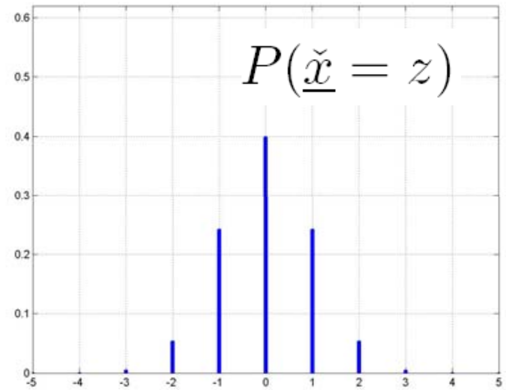
$$f_{\check{e}_0}(\epsilon) = \sum_{z \in \mathbb{Z}^n} f_{\check{e}_0|\hat{x}=z}(\epsilon) P(\underline{\check{x}} = z)$$

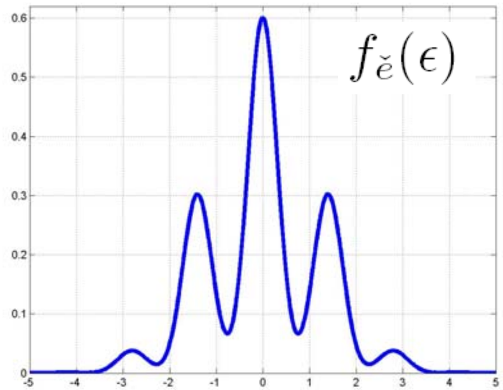
with

$$f_{\check{e}_0|\hat{x}=z}(\epsilon) \sim \mathrm{N}(A_{0|y}(x - z), Q_{y_0 y_0|y})$$

$$P(\underline{\check{x}} = z) = \int_{S_z} f_{\hat{x}}(x)dx$$







Model validation

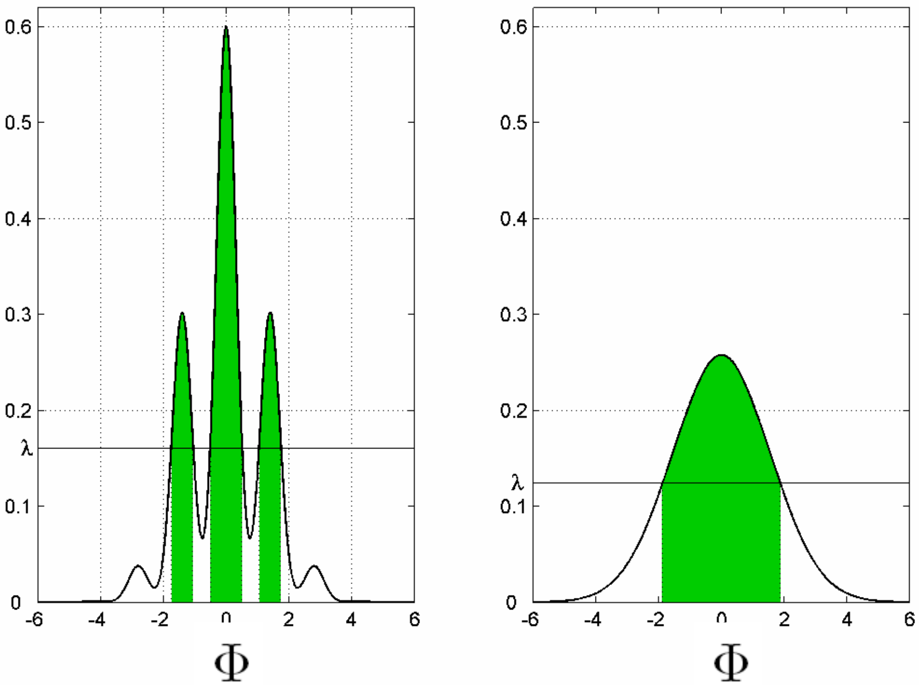
Validation test: Accept if $\breve{e}_0 \in \Omega$, otherwise reject

Definition of optimal acceptance region:

$$\min_{\Omega} \mathrm{Vol}(\Omega) \quad \text{subject to} \quad P(\breve{e}_0 \in \Omega) = 1 - \alpha$$

Optimal acceptance region:

$$\Phi = \{\breve{e}_0 \in \mathbb{R}^{m_0} |\ f_{\breve{e}_0}(\breve{e}_0) > \lambda_\alpha\}$$



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Validation: Kalman filter example

prediction error = predicted residual = innovation

Real state: $\underline{v}_k = \underline{y}_k - A_k \underline{\hat{x}}_{k|k-1}$ $\qquad \underline{v}_k \sim \mathrm{N}(0, Q_{v_k v_k})$

Local Detection: Accept if $\underline{v}_k^T Q_{v_k v_k}^{-1} \underline{v}_k \leq \chi^2_\alpha(m_k, 0)$

Integer state: $\underline{v}'_k = \underline{y}_k - A_k \underline{\check{x}}_{k|k-1}$ $\qquad f_{v'_k}(v) = \sum_{z \in \mathbb{Z}^n} f_{v_k|x_k=z}(v) P(\underline{\check{x}}_{k|k-1} = z)$

Local Detection: Accept if $f_{v'_k}(v) \geq \lambda_\alpha$

Acceptance region for local detection in $\mathbb{R}^2$

Conclusions

- **Extended LS-Prediction (Collocation) so as to include integer parameters**
- **Obtained distributional results of predictors and their errors**
- **Distributional results make rigorous model validation possible**