# Ellipsoidal Geometry

# and

# Conformal Mapping

by

KAI BORRE

Preliminary version    March 2001

## Foreword

Most geodetically oriented textbooks on ellipsoidal geometry and conformal mapping are written in the German language. This has motivated me to compile a useful English text for students who follow the English M.Sc. programme in "GPS Technology" at Aalborg University.

The main bulk of the present text is taken from my Danish textbook "Landmåling". However the Gauss mid-latitude formulas for solving the geodetic problems have been substituted by Bessel's solution. The latter is a little more evolved but we arrive at formulas valid for points thousands of kilometers apart.

A good textbook of differential geometry that supports and augments the present text is Struik (1988). I also expect some students will be happy with Guggenheimer (1977) when dealing with specialized topics.

Fjellerad, March 2001

*Kai Borre*

# Contents

# 1

# Geometry of the Ellipsoid

ABSTRACT We investigate the geometrical properties of the ellipsoid (by ellipsoid we *always* mean an ellipsoid of revolution), globally as well as locally. On this curved surface we study the basic transformation between polar and curvilinear coordinates. The series expansions contain a sufficient number of terms to guarantee computational accuracies at mm level.

We start by deriving the length of the meridian arc from Equator to a point with arbitrary latitude. This computation is fundamental to all conformal mappings. Next we deal with Bessel's formulas for the direct and reverse geodetic problems. Finally the transformation between 3-D Cartesian and ellipsoidal coordinates is dealt with. This set of formulas is essential for positioning with GPS.

## 1.1 The Ellipsoid of Revolution

All surveying observations are refered to an ellipsoid of revolution. This surface is described by rotating an ellipse about its minor axis. The equation of the ellipse in a $p$, $Z$-coordinate system is, see Figure 1.1

$$\frac{p^2}{a^2} + \frac{Z^2}{b^2} = 1. \tag{1.1}$$

This *meridian ellipse* is determined by its semi-major $a$ and semi-minor $b$ axes. However, more often the dimensionless quantities flattening $f$, first eccentricity $e$, and second eccentricity $e'$ are used. For later reference we give a complete list of the ties among all these

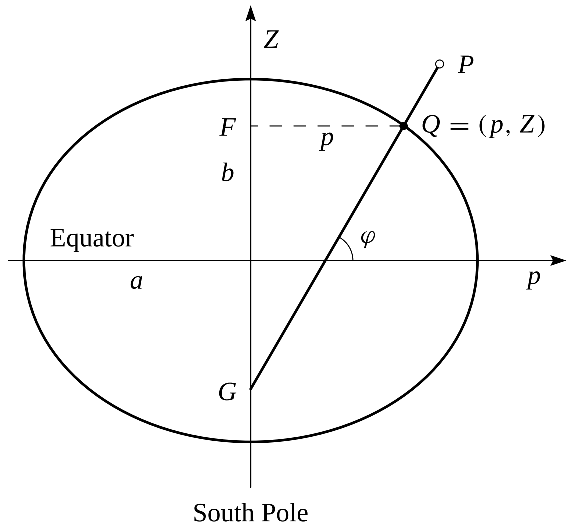


**Figure 1.1** Meridian ellipse

quantities (it is a good exercise to verify a few of the equalities):

$$
\begin{aligned}
\frac{a-b}{a} &= f &&= 1-\sqrt{1-e^2} &&= 1-\frac{1}{\sqrt{1+e'^2}}\\
\frac{a^2-b^2}{a^2} &= f(2-f) &&= e^2 &&= \frac{e'^2}{1+e'^2}\\
\frac{a^2-b^2}{b^2} &= \frac{f(2-f)}{(1-f)^2} &&= \frac{e^2}{1-e^2} &&= e'^2\\
\frac{a-b}{a+b} &= \frac{f}{2-f} &&= \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}} &&= \frac{\sqrt{1+e'^2}-1}{\sqrt{1+e'^2}+1}\\
\frac{a^2-b^2}{a^2+b^2} &= \frac{f(2-f)}{1+(1-f)^2} &&= \frac{e^2}{2-e^2} &&= \frac{e'^2}{2+e'^2}\\
\frac{b}{a}=\frac{a}{c} &= 1-f &&= \sqrt{1-e^2} &&= \frac{1}{\sqrt{1+e'^2}}.
\end{aligned}
\tag{1.2}
$$

Furthermore, we also use the radius of curvature $c = a^2/b$ at the poles:

$$c = \frac{a^2}{b} = a\sqrt{1+e'^2} = b\bigl(1+e'^2\bigr). \tag{1.3}$$

Most often a reference ellipsoid is given by $a$ and $f$. Table 1.1 quotes values for frequently used ellipsoids. The fattening has a very small value $f = 0.003\,35$. A globe of radius $a = 1$ m would only be flattened by 7 mm. In this case $e' \approx 0.082\,09$.

A point $Q$ on the ellipsoid is determined by (latitude, longitude) $= (\varphi, \lambda)$. The *geographical latitude* $\varphi$ is the angle between the normal at $Q$ and the plane of the Equator. For an ellipse, the normal at $Q$ does *not* contain the origin. (This is the price we must pay

**Table 1.1** Parameters for reference ellipsoids

| Name | $a$ [m] | $f = (a - b)/a$ |
|---|---|---|
| International 1924 | 6 378 388 | 1/297 |
| Krassovsky 1948 | 6 378 245 | 1/298.3 |
| International 1980 | 6 378 137 | 1/298.257 |
| World Geodetic System 1984 | 6 378 137.0 | 1/298.257 223 563 |

for flattening; $\varphi$ is not the angle from the center of the ellipse.) The *geographical longitude* $\lambda$ is the angle between the plane of the meridian of $Q$ and the plane of a reference meridian (through Greenwich).

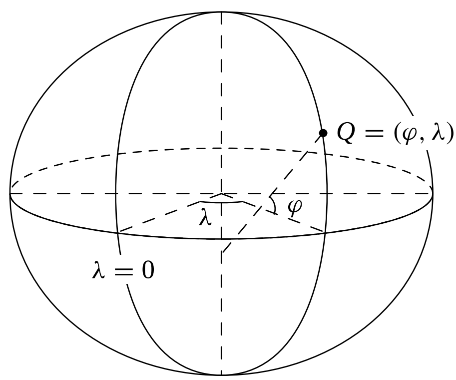


**Figure 1.2** The parameter system of the ellipsoid

If we connect all points on the ellipsoid with equal latitude $\varphi$ we get a closed curve with $\varphi =$ constant; it is called a *parallel*. In a similar way all points with equal longitude lie on the parameter curve $\lambda =$ constant, which is a *meridian*. Meridians and parallels constitute the geographical net (or grid). The meridians are ellipses while the parallels are circles. The geographical coordinates can be taken as angles as well as surface coordinates.

Geographical longitude is reckoned from the meridian $\lambda_0 = 0$ of Greenwich. A surface curve through $Q$ makes an angle with the meridian; this angle is called the *azimuth* $A$ of the curve. The azimuth of a parallel is $A = \pi/2$ or $3\pi/2$. Normally $A$ is reckoned clockwise from the northern branch of the meridian.

## 1.2 Principal Curvatures

Planes that contain the surface normal at $P$ are called ***normal planes***. In order to investigate the curvature of a surface at a point $P$ we intersect the surface with normal planes with various azimuths.

Next the radius of curvature $R$ of the intersecting curves, normal sections, between the surface and the normal planes can be determined. The two normal sections having the largest $R_1$ and the smallest $R_2$ values are of special interest. Those normal sections are orthogonal.

The socalled *Gaussian curvature* $K = (R_1 R_2)^{-1}$ is often used to define the *Gaussian osculating sphere* with radius $R = \sqrt{R_1 R_2}$.

The *curvature* $\kappa$ of a normal section making the angle $\alpha$ with the principal curvature plane belonging to $R_1$ is given by *Euler's formula*

$$\kappa = \frac{\cos^2\alpha}{R_{\min}} + \frac{\sin^2\alpha}{R_{\max}}. \tag{1.4}$$

A plane that does not contain the surface normal is inclined. The radius of curvature for an inclined plane equals the radius of curvature in the corresponding normal section $R$ times cosine of the angle $\theta$ between the planes. This is called *Meusnier's theorem*

$$\rho = R\cos\theta. \tag{1.5}$$

To determine the principal curvature for the normal sections of the ellipsoid we only need a few simple geometrical considerations. Any meridian plane is a plane of symmetry, so the direction of the meridian is a direction of principal curvature. We call the radius of curvature $M$ in this direction and the radius of curvature in the orthogonal direction is called $N$. The ellipsoid is flattened at the poles and accordingly the curvature at any point is larger in the direction of the meridian so $M$ is smaller than $N$. $N$ is also called the *prime vertical radius of curvature*.

Euler's and Meusnier's formulas give full information concerning the curvature of any curve through $Q$ on the surface.

From Figure 1.3 we have $dB = M\,d\varphi$ or $M = dB/d\varphi$. We want to express the length of arc $dB$ as a function of latitude $\varphi$ and start from equation (1.1) for the ellipse. The relation between latitude $\varphi$ and the Cartesian coordinates $(p, Z)$ of a point of the ellipse is given as

$$\tan\varphi = -\frac{dp}{dZ}. \tag{1.6}$$

From (1.1) we may express $p$ explicitly as a function of $Z$. We differentiate and insert into (1.6)

$$\tan\varphi = \left(\frac{a}{b}\right)^2 \frac{Z}{p}. \tag{1.7}$$

We want to eliminate $Z$ from (1.7) and insert from (1.1) and use (1.2) and get

$$p = \frac{a\cos\varphi}{\sqrt{\cos^2\varphi + \frac{b^2}{a^2}\sin^2\varphi}} = \frac{a\sqrt{1+e'^2}\cos\varphi}{\sqrt{1+e'^2\cos^2\varphi}}. \tag{1.8}$$

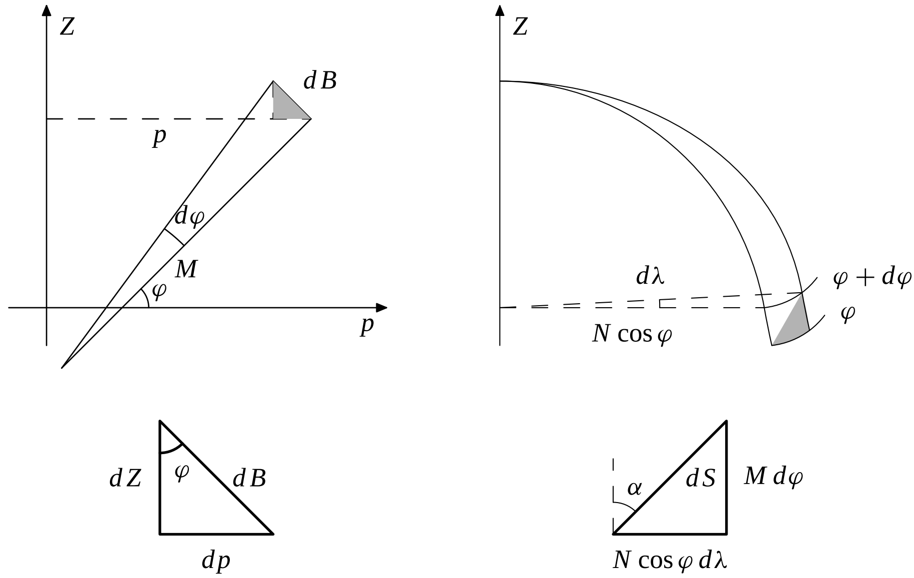


**Figure 1.3** Left: The differential triangle for the meridian with an increase $d\varphi$ in latitude. Right: The differential triangle belonging to an increase $d\lambda$ in longitude. Below: The shaded figures are shown in an enlarged scale

According to Figur 1.3 we have $dB = -dp/\sin\varphi$ and get by insertion into (1.8)

$$\begin{aligned}M &= \frac{dB}{d\varphi} = -\frac{1}{\sin\varphi}\frac{dp}{d\varphi}\\ &= -\frac{1}{\sin\varphi}\Big(\sqrt{1+e'^2\cos^2\varphi}\big(-a\sin\varphi\sqrt{1+e'^2}\big)\\ &\qquad + a\sqrt{1+e'^2}\big(1+e'^2\cos^2\varphi\big)^{-1/2}e'^2\cos^2\varphi\sin\varphi\Big)\Big/\big(1+e'^2\cos^2\varphi\big)\\ &= \frac{c}{(1+e'^2\cos^2\varphi)^{3/2}}.\end{aligned} \tag{1.9}$$

***Auxiliary variables*** We start by defining the variables $V$ and $\eta$:

$$V^2 = 1 + \eta^2 = 1 + e'^2\cos^2\varphi. \tag{1.10}$$

It is important to remember that both $V$ and $\eta$ are functions of $\varphi$. Next

$$W^2 = 1 - e^2\sin^2\varphi \tag{1.11}$$

and finally the important ***reduced latitude*** $\beta$:

$$\tan\beta = (1 - f)\tan\varphi. \tag{1.12}$$

This can be rewritten as

$$\left.\begin{aligned}\sin\varphi &= V\sin\beta\\ \cos\varphi &= W\cos\beta\end{aligned}\right\}. \tag{1.13}$$

We collect many useful relations:

$$\begin{aligned}
\frac{\cos\varphi}{\cos\beta} &= \sqrt{1-e^2\sin^2\varphi} = \frac{1}{\sqrt{1+e'^2\sin^2\beta}} = W = \frac{V}{\sqrt{1+e'^2}}\\
\frac{\sin\varphi}{\sin\beta} &= \sqrt{1+e'^2\cos^2\varphi} = \frac{1}{\sqrt{1-e^2\cos^2\beta}} = \frac{W}{\sqrt{1-e^2}} = V\\
\frac{d\varphi}{d\beta} &= \frac{W^2}{1-e^2} = \frac{V^2}{\sqrt{1+e'^2}}\\
\frac{\tan\beta}{\tan\varphi} &= \frac{b}{a} = 1-f = \sqrt{1-e^2} = \frac{1}{\sqrt{1+e'^2}}.
\end{aligned} \tag{1.14}$$

Now (1.9) can be expressed as

$$M = \frac{c}{V^3} \qquad \text{or} \qquad M = \frac{a(1-e^2)}{(1-e^2\sin^2\varphi)^{3/2}}. \tag{1.15}$$

In order to calculate $N$ we consider at point $Q$ both the circle of the parallel and the circle of curvature. They lie in different planes the traces of which in Figure 1.1 are the lines $QG$ and $QF$. The line section $QF = p$ is radius for the parallel and $N$ is the radius of curvature in the normal section at $Q$; they make the angle $\varphi$ and according to the theorem of Meusnier (1.5) we get

$$p = N\cos\varphi. \tag{1.16}$$

Comparing with (1.8) we get

$$N = \frac{a\sqrt{1+e'^2}}{\sqrt{1+e'^2\cos\varphi}} = \frac{c}{V}$$

and using (1.14) once again we get

$$N = \frac{c}{V} = c\sqrt{1+e'^2\sin^2\beta}. \tag{1.17}$$

Eliminating $c$ from (1.15) and (1.17) yields

$$V^2 = \frac{N}{M}. \tag{1.18}$$

The *Gaussian curvature $K$ for the ellipsoid* is

$$K = \frac{1}{\sqrt{MN}} = \frac{V^2}{c}.$$

The radius of curvature at the poles are $M_{90} = N_{90} = c = a^2/b$. The radius of curvature at Equator is $M_0 = b^2/a$ and $N_0 = a$. The radius of curvature in a direction of azimuth $\alpha$ is according to (1.4)

$$\frac{1}{R_A} = \frac{\cos^2\alpha}{M} + \frac{\sin^2\alpha}{N} = \frac{1+\eta^2\cos^2\alpha}{N}. \tag{1.19}$$

In subsequent sections we often need derivatives of $V$ and $\eta^2$ as defined in (1.10). Once and for all we calculate their derivatives with respect to $\varphi$:

$$\frac{d\eta^2}{d\varphi} = e'^2 2\cos\varphi(-\sin\varphi) = -2\eta^2\tan\varphi. \tag{1.20}$$

Following geodetic tradition, we introduce

$$t = \tan\varphi \tag{1.21}$$

and get

$$\frac{d\eta^2}{d\varphi} = -2\eta^2 t \tag{1.22}$$

$$\frac{d^2\eta^2}{d\varphi^2} = -2\eta^2(1-t^2) \tag{1.23}$$

$$\frac{d^3\eta^2}{d\varphi^3} = 8\eta^2 t. \tag{1.24}$$

Similarly starting from (1.10) we get

$$\frac{dV}{d\varphi} = -\frac{\eta^2 t}{V} \tag{1.25}$$

$$\frac{d^2V}{d\varphi^2} = -\frac{\eta^2}{V^3}(1-t^2+\eta^2) \tag{1.26}$$

$$\frac{d^3V}{d\varphi^3} = \frac{\eta^2 t}{V^5}(4+5\eta^2+3\eta^2t^2+\eta^4). \tag{1.27}$$

## 1.3 Length of a Meridional Arc

When dealing with conformal mapping equations we often need to compute the length of the arc of a meridian between two points with latitudes $\varphi_1$ and $\varphi_2$. From (1.15) we have

$$B_{12} = \int_{\varphi_1}^{\varphi_2} M\,d\varphi = c\int_{\varphi_1}^{\varphi_2} \frac{d\varphi}{V^3}.$$

This integration leads to an elliptic integral of the second kind; from a computational point of view this is no useful path to follow. Instead we develop the denominator into a series

$$V^{-3} = 1 - \frac{3}{2}e'^2\cos^2\varphi + \frac{15}{8}e'^4\cos^4\varphi - \frac{35}{16}e'^6\cos^6\varphi + - \cdots$$

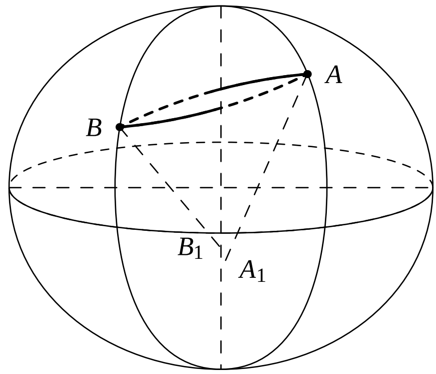


**Figure 1.4** Forward and reverse normal sections

and integrate term-wise. For the international ellipsoid 1924 with $c = 6\,399\,936.608\,1$ m and $e'^2 = 0.006\,768\,170$ the arc length $B$ in meters from Equator $\varphi_1 = 0$ to the latitude $\varphi_2 = \varphi$, in radians, becomes

$$B = 6\,367\,654.500\,\varphi - 32\,282.118\,64 \sin\varphi \cos\varphi \times (1 - 0.004\,228\,875 \cos^2\varphi + 0.000\,022\,402 \cos^4\varphi). \quad (1.28)$$

**Example 1.1** Given $\varphi = 56°$, find $B$.

$$B = 6\,223\,646.055 - 14\,965.729\,60(1 - 0.001\,322\,355 + 0.000\,002\,190)$$
$$= 6\,208\,700.083\,\text{m}.$$

## 1.4 Normal Section and the Geodesic

All normals to the surface of a sphere contain the center of the sphere. A normal plane at point $A$ containing point $B$ also contains the surface normal at $B$. However on the ellipsoid the surface normals at $A$ and $B$ only intersect if $A$ and $B$ lie on the same meridian or the same parallel. In all other cases the surface normals are skew to each other.

A normal plane at point $A$ containing point $B$ does not contain the surface normal at $B$ and the normal plane at $B$ containing point $A$ does not contain the surface normal at $A$. Both planes incline to each other. They cut the surface of the ellipsoid in different curves. This fact is illustrated in Figure 1.4. The points $AA_1B$ define the normal plane of $A$ and the points $BB_1A$ define the normal plane at $B$. The forward and the reverse sight over the same side do not coincide and we have to define which curve shall make the basis for our computations.

It would be natural to choose one of the *normal sections*; but the formulas become complicated. In the 1700-years one often used the secant *AB*. But since the publication Gauss (1827) appeared one only uses the *geodesic* which is a curve lying between the forward and the reverse normal sections, see Figure 1.5.

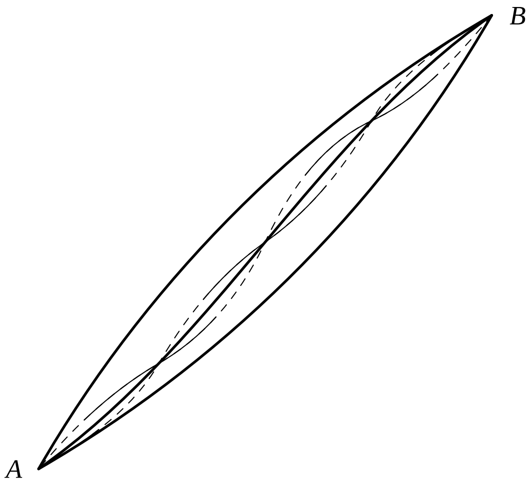


**Figure 1.5** The lower heavy curve is the normal section from $A$ to $B$, the upper heavy curve is the normal section from $B$ to $A$, and the heavy curve in between is the geodesic between $A$ and $B$. It is the same in both directions. We assume to be on the northern hemisphere. The thin lines indicate short normal sections as defined by using a theodolite for setting out a straight line between $A$ and $B$. If the number of setups tends to infinity and the distance between the setup points tends to zero the short normal sections will collaps to a single curve, viz. the geodesic

Most trigonometrical computations on the surface of the ellipoisd are based on the geodesic. Therefore we must be familiar with its geometrical and analytical characteristics. Espcially we have to know the relationship between the geodesic and the normal sections as we use normal sections when observing with the theodolite. The appropriate investigation starts from the differential equations that characterize the geodesic in the ellipsoidal parametric system $(\varphi, \lambda)$.

The derivations lead to comprehensive mathematical manipulations and for the moment we limit ourselves to summarizing the main results:

- The binormal of the geodesic coincides with the surface normal
- A geodesic is the shortest curve between two points on the surface, and
- A geodesic on a surface $S$ is a curve on $S$ whose geodesic curvature is identically zero.

The difference in azimuth between the geodesic and the normal section is

$$\Delta\alpha \approx -\frac{\eta_A^2}{6N_A^2} S^2 \sin\alpha_A \cos\alpha_A \tag{1.29}$$

and particularly for the international ellipsoid

$$\Delta\alpha_{[\prime\prime]} \approx -2.8'' \cos^2\varphi_A \sin 2\alpha_A S^2_{[\mathrm{km}]} \cdot 10^{-6}.$$

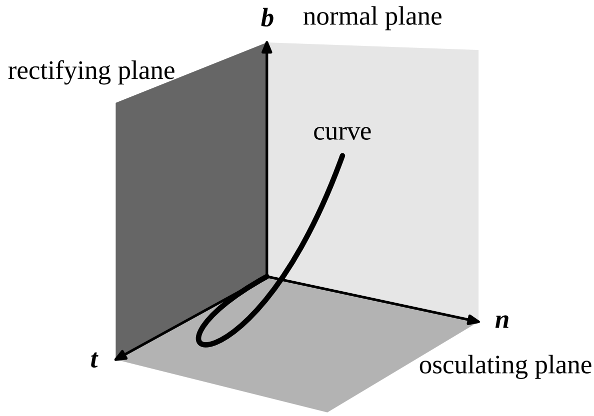


**Figure 1.6** Trihedron spanned by the tangent vector $\boldsymbol{t}$, the normal vector $\boldsymbol{n}$, and the binormal vector $\boldsymbol{b}$

With $S = 100\,\text{km}$, $\alpha_A = 45°$, and $\varphi_A = 57°$ we get $\Delta\alpha = -0.008''$. This quantity is small compared to the accuracy of observation.

The difference in length between the geodesic and the normal section is

$$\Delta S = \frac{\eta_A^4}{90 N_A^4} S^5 \sin^2\alpha_A \cos^2\alpha_A.$$

With $S = 100\,\text{km}$, $\alpha_A = 45°$, and $\varphi_A = 57°$ we get $\Delta S = 7 \cdot 10^{-8}\,\text{mm}$. This quantity can be ignored in all cases.

## 1.5 Frenet's Formulas for a Geodesic

We start by studying the differential equations for the geodesic in general and next specialize them to a surface of revolution.

Most often modern differential geometry is abstract; but we need a very concrete derivation and choose to use vector analysis for so doing. A good, general exposition of differential geometry can be found in Struik (1988); for the specific circumstances about the geodesic we follow Lehn (1957).

A ***curve*** may be represented by the equation $\boldsymbol{r} = \boldsymbol{r}(s)$ and $\boldsymbol{R}$ will denote the position vector of a current point in space not necessarily lying on the curve. The parameter $s$ denotes arc length. Let $\gamma$ be a curve with second order derivatives and $P$, $Q$ be two neighbouring points on $\gamma$. Then the limiting position as $Q \to P$ of that plane which contains the tangent line at $P$ and the point $Q$ is called the ***osculating plane*** of $\gamma$ at $P$.

Assume that $\gamma$ is parametrized with respect to arc length $s$ and that the parameters of $P$, $Q$ are 0 and $s$ respectively. The equation of the plane through the tangentline at $P$ and the point $Q$ is

$$\begin{vmatrix} \boldsymbol{R} - \boldsymbol{r}(0) & \boldsymbol{r}'(0) & \boldsymbol{r}(s) - \boldsymbol{r}(0) \end{vmatrix} = 0.$$

It is convenient to denote differentiation with respect to arc length $s$ by a prime; with this convention we get

$$\boldsymbol{r}(s) - \boldsymbol{r}(0) = s\boldsymbol{r}'(0) + \tfrac{1}{2}s^2\boldsymbol{r}''(0) + o(s^2)$$

and the equation of the osculating plane is

$$\left|\boldsymbol{R} - \boldsymbol{r}(0) \quad \boldsymbol{r}'(0) \quad \boldsymbol{r}''(0)\right| = 0.$$

The unit tangent vector $\boldsymbol{t}$ becomes

$$\boldsymbol{t} = \boldsymbol{r}'. \tag{1.30}$$

The vector

$$\boldsymbol{t}' = \boldsymbol{r}'' \tag{1.31}$$

lies in the osculating plane and is also normal to $\boldsymbol{t}$. A unit vector along the principal normal is denoted by $\boldsymbol{n}$ and the *curvature* by $\kappa$ and we have

$$\boldsymbol{t}' = \kappa\boldsymbol{n}. \tag{1.32}$$

The *binormal line* at $P$ is the normal in a direction orthogonal to the osculating plane. The sense of the unit vector $\boldsymbol{b}$ along the binormal is chosen so that the triad $\boldsymbol{t}$, $\boldsymbol{n}$, and $\boldsymbol{b}$ forms a right-handed system of axes:

$$\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}. \tag{1.33}$$

The relations

$$\boldsymbol{t}' = \kappa\boldsymbol{n} \tag{1.34}$$

$$\boldsymbol{n}' = -\kappa\boldsymbol{t} + \tau\boldsymbol{b} \tag{1.35}$$

$$\boldsymbol{b}' = -\tau\boldsymbol{n} \tag{1.36}$$

are known as the ***Frenet formulas*** and they underlie many investigations in the theory of curves and surfaces. The reader is well advised to commit them to memory.

The first and the third relations have already been obtained. The second relation follows from differentiating the identity $\boldsymbol{n} = \boldsymbol{b} \times \boldsymbol{t}$ to get

$$\boldsymbol{n}' = -\tau\boldsymbol{n} \times \boldsymbol{t} + \boldsymbol{b} \times \kappa\boldsymbol{n} = \tau\boldsymbol{b} - \kappa\boldsymbol{t}.$$

The ***curvature*** $\kappa$ is

$$\kappa = \left|\boldsymbol{r}' \times \boldsymbol{r}''\right|. \tag{1.37}$$

The ***torsion*** $\tau$ is

$$\tau = \frac{\left|\boldsymbol{r}' \quad \boldsymbol{r}'' \quad \boldsymbol{r}'''\right|}{\kappa^2}. \tag{1.38}$$

Let the curve $\boldsymbol{r}$ lie on a sufficiently smooth *surface* in space. The surface is given by two independent parameters $u$ and $v$:

$$x = f(u, v)$$
$$y = g(u, v)$$
$$z = h(u, v).$$

A point $P$ on the surface may then be represented by the vector equation

$$\boldsymbol{r} = \boldsymbol{r}(u, v). \tag{1.39}$$

For the ***curve on the surface*** we may define another moving rigt-handed triad of unit vectors. We call the unit tangent vector to the curve $\boldsymbol{w}_1$. The unit vector $\boldsymbol{w}_3$ normal to the surface is given as

$$\boldsymbol{w}_3 = \frac{\boldsymbol{r}'_u \times \boldsymbol{r}'_v}{\left|\boldsymbol{r}'_u \times \boldsymbol{r}'_v\right|}. \tag{1.40}$$

The direction of $\boldsymbol{w}_3$ can be chosen freely. Finally $\boldsymbol{w}_2$ is defined by the vector equation

$$\boldsymbol{w}_3 = \boldsymbol{w}_1 \times \boldsymbol{w}_2. \tag{1.41}$$

Hence this vector lies in the tangent plane for the surface and is perpendicular to the tangent. Setting (tangent, normal, binormal) = $(\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3)$ we have

$$\boldsymbol{w}'_1 = \kappa_g \boldsymbol{w}_2 + \kappa_n \boldsymbol{w}_3 \tag{1.42}$$
$$\boldsymbol{w}'_2 = -\kappa_g \boldsymbol{w}_1 + \tau_g \boldsymbol{w}_3 \tag{1.43}$$
$$\boldsymbol{w}'_3 = -\kappa_n \boldsymbol{w}_1 - \tau_g \boldsymbol{w}_2. \tag{1.44}$$

The curvature $\kappa_n$ of the normal section is

$$\kappa_n = \boldsymbol{w}_3 \cdot \boldsymbol{r}''. \tag{1.45}$$

The geodetic curvature $\kappa_g$ is

$$\kappa_g = \left|\boldsymbol{w}_3 \quad \boldsymbol{r}' \quad \boldsymbol{r}''\right|. \tag{1.46}$$

The geodetic torsion $\tau_g$ is

$$\tau_g = \left|\boldsymbol{w}_1 \quad \boldsymbol{w}_3 \quad \boldsymbol{w}'_3\right|. \tag{1.47}$$

For any curve we have

$$\boldsymbol{r}' \equiv \boldsymbol{t} \equiv \boldsymbol{w}_1. \tag{1.48}$$

It follows that $\boldsymbol{r}' \cdot \boldsymbol{r}' = 1$ and by differentiation

$$\boldsymbol{r}' \cdot \boldsymbol{r}'' = 0. \tag{1.49}$$

Next we specialize the ***curve to be a geodesic***, i.e. the curve has zero geodetic curvature

$$\kappa_g = \left|\boldsymbol{w}_3 \quad \boldsymbol{r}' \quad \boldsymbol{r}''\right| = 0. \tag{1.50}$$

Then the Frenet formulas reduce to

$$\boldsymbol{w}'_1 = \kappa_n \boldsymbol{w}_3 \tag{1.51}$$
$$\boldsymbol{w}'_2 = \tau_g \boldsymbol{w}_3 \tag{1.52}$$
$$\boldsymbol{w}'_3 = -\kappa_n \boldsymbol{w}_1 - \tau_g \boldsymbol{w}_2. \tag{1.53}$$

Remember $\boldsymbol{t} \equiv \boldsymbol{w}_1$, and comparing with (1.34)–(1.36), and choosing $\boldsymbol{n} = \boldsymbol{w}_3$ we obtain $\kappa = \kappa_n$. Hence we have

$$\frac{d\boldsymbol{n}}{ds} = \frac{d\boldsymbol{w}_3}{ds} = -\kappa\boldsymbol{t} + \tau\boldsymbol{b} = -\kappa_n\boldsymbol{w}_1 - \tau_g\boldsymbol{w}_2$$

or

$$\tau\boldsymbol{b} = -\tau_g\boldsymbol{w}_2.$$

Also

$$\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n} = \boldsymbol{w}_1 \times \boldsymbol{w}_3 = -\boldsymbol{w}_2$$

and we conclude

$$\tau = \tau_g \qquad \text{and} \qquad \boldsymbol{b} = -\boldsymbol{w}_2. \tag{1.54}$$

So the ***Frenet formulas for a geodesic on a surface*** are

$$\boldsymbol{t}' = \kappa_n\boldsymbol{n} \tag{1.55}$$

$$\boldsymbol{n}' = -\kappa_n\boldsymbol{t} + \tau_g\boldsymbol{b} \tag{1.56}$$

$$\boldsymbol{b}' = -\tau_g\boldsymbol{n}. \tag{1.57}$$

## 1.6 Clairaut's Equation

In this section we derive Clairaut's equation which is the starting point for Bessel's solution of the two main geodetic problems. Clairaut's equation is not found in standard textbooks on differential geometry. It deals with the geodesic on a surface of rotation.

We find an equation that characterizes a geodesic on a *surface of revolution*, with the $z$-axis as axis of revolution. Let the surface be given as

$$F = x^2 + y^2 - \psi(z) = 0.$$

A normal vector $\boldsymbol{n}$ to this surface is

$$\boldsymbol{n} = \begin{bmatrix} \dfrac{\partial F}{\partial x} \\ \dfrac{\partial F}{\partial x} \\ \dfrac{\partial F}{\partial x} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ -\psi'(z) \end{bmatrix}.$$

Remember (1.50) with $\boldsymbol{w}_3 = \boldsymbol{n}$:

$$\begin{vmatrix} \boldsymbol{n} & \boldsymbol{r}' & \boldsymbol{r}'' \end{vmatrix} = 0$$

that is $\boldsymbol{n}$ lies in the osculating plane. Also $\boldsymbol{n}$ is perpendicular to $\boldsymbol{r}' = \boldsymbol{t}$. Remember (1.49), so $\boldsymbol{r}''$ is perpendicular to $\boldsymbol{r}'$. Hence $\boldsymbol{n}$, $\boldsymbol{r}'$, and $\boldsymbol{r}''$ all lie in the osculating plane and as $\boldsymbol{n}$ and $\boldsymbol{r}''$ both are perpendicular to $\boldsymbol{r}'$ they must be parallel which implies

$$\boldsymbol{n} \times \boldsymbol{r}'' = \boldsymbol{0}.$$

We focus on the third component of this equation:

$$\begin{vmatrix} x & y \\ x'' & y'' \end{vmatrix} = 0$$

or

$$xy'' - yx'' = 0$$

or after integration

$$xy' - yx' = \text{constant}. \tag{1.58}$$

We repeat the formulas (1.118)–(1.120):

$$\boldsymbol{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} N\cos\varphi\cos\lambda \\ N\cos\varphi\sin\lambda \\ (1+e'^2)^{-1}N\sin\varphi \end{bmatrix}. \tag{1.59}$$

In the last line we have made a small rewriting using (1.2). Recall the basic equation (1.16): $p = N\cos\varphi$ and we have

$$x = p\cos\lambda, \qquad x' = \frac{dp}{ds}\cos\lambda - p\sin\lambda\frac{d\lambda}{ds} \tag{1.60}$$

$$y = p\sin\lambda, \qquad y' = \frac{dp}{ds}\sin\lambda + p\cos\lambda\frac{d\lambda}{ds}. \tag{1.61}$$

Insertion into (1.58) yields

$$p^2\frac{d\lambda}{ds} = \text{constant}. \tag{1.62}$$

Our next big derivation is to determine $\boldsymbol{n}$, $\boldsymbol{t}$, $\boldsymbol{b}$ from $\boldsymbol{r}$. We start by bringing some useful formulas together. From the lower right part of Figure 1.3 and (1.18) we get

$$\frac{d\varphi}{ds} = \frac{\cos\alpha}{M} = \frac{V^2\cos\alpha}{N} \tag{1.63}$$

$$\frac{d\lambda}{ds} = \frac{\sin\alpha}{N\cos\varphi}. \tag{1.64}$$

Now we insert (1.64) into (1.62) and get

$$N\cos\varphi\sin\alpha = \text{constant} \tag{1.65}$$

which is ***Clairaut's equation***: On a surface of revolution the product of the sine of the angle of the geodesic and meridian with the distance from the axis of revolution is constant. Using geographical latitude $\varphi$ this is

$$a\sin\alpha_0 = N\cos\varphi\sin\alpha = \text{constant} \tag{1.66}$$

or using reduced latitude $\beta$ on the unit sphere

$$\sin\alpha_0 = \cos\beta\sin\alpha = \cos\beta_{\max} = \text{constant}. \tag{1.67}$$

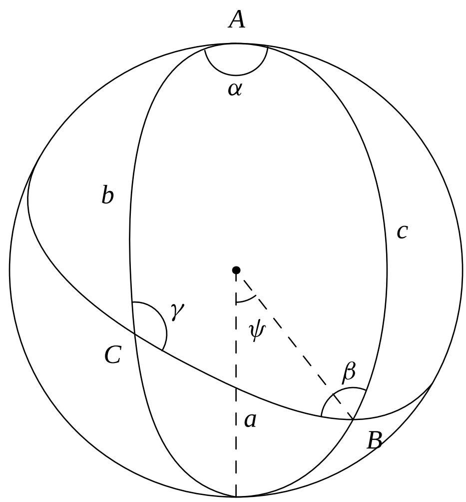


**Figure 1.7** Sine theorem of spherical trigonometry

**Example 1.2** On a unit sphere $r = \sin\psi$. We take a spherical triangle with one vertex at the north pole and the two other vertices on meridians from the north to the south pole. In the notation of Figure 1.7, $r(C) = \sin b$, $r(B) = \sin c$, and for the arc of the great circle joining $B$ to $C$, $\theta(C) = \gamma$ and $\theta(B) = \beta$. Hence $\sin b \sin\gamma = \sin c \sin\beta$ or

$$\frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma} = \frac{\sin a}{\sin\alpha}. \tag{1.68}$$

Clairaut's theorem on the sphere is the sine theorem of spherical trigonometry. □

In the following calculations we need to know $\frac{dN^{-1}}{ds}$ and $\frac{dN}{ds}$. We start by differentiating (1.10)

$$\frac{dV^2}{d\varphi} = -2e'^2 \sin\varphi\cos\varphi. \tag{1.69}$$

Next we use (1.63):

$$\frac{dV^2}{ds} = \frac{dV^2}{d\varphi}\frac{d\varphi}{ds} = -2e'^2 \sin\varphi\cos\varphi \frac{V^2\cos\alpha}{N} = -\frac{2V^2}{N}e'^2\sin\varphi\cos\varphi\cos\alpha.$$

Now $\frac{dV^2}{ds} = 2V\frac{dV}{ds}$ and we have

$$\frac{dV}{ds} = -\frac{V}{N}e'^2\sin\varphi\cos\varphi\cos\alpha.$$

Recall from (1.17) that $V = cN^{-1}$ and we get

$$\frac{dV}{ds} = c\frac{dN^{-1}}{ds} = -V^2e'^2\sin\varphi\cos\varphi\cos\alpha.$$

Again $\frac{dN}{ds} = -N^2 \frac{dN^{-1}}{ds}$ and

$$\frac{dN}{ds} = e'^2 \sin\varphi \cos\varphi \cos\alpha. \tag{1.70}$$

In the subsequent derivations we need (1.59) and (1.63) and (1.64) and (1.70). We indicate the calculation for $x'$:

$$\begin{aligned} x' = \frac{dx}{ds} &= \cos\varphi \cos\lambda \frac{dN}{ds} - N \sin\varphi \cos\lambda \frac{d\varphi}{ds} - N \cos\varphi \sin\lambda \frac{d\lambda}{ds} \\ &= -\cos\alpha \sin\varphi \cos\lambda - \sin\alpha \sin\lambda. \end{aligned}$$

The total result is

$$\boldsymbol{r}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -\cos\alpha \sin\varphi \cos\lambda - \sin\alpha \sin\lambda \\ -\cos\alpha \sin\varphi \sin\lambda + \sin\alpha \cos\lambda \\ \cos\alpha \cos\varphi \end{bmatrix}. \tag{1.71}$$

Evidently

$$\boldsymbol{n} = \begin{bmatrix} -\cos\varphi \cos\lambda \\ -\cos\varphi \sin\lambda \\ -\sin\varphi \end{bmatrix} \quad \text{and} \quad \boldsymbol{t} = \boldsymbol{r}' = \begin{bmatrix} -\cos\alpha \sin\varphi \cos\lambda - \sin\alpha \sin\lambda \\ -\cos\alpha \sin\varphi \sin\lambda + \sin\alpha \cos\lambda \\ \cos\alpha \cos\varphi \end{bmatrix}$$

and

$$\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n} = \begin{bmatrix} -\sin\alpha \sin\varphi \cos\lambda + \cos\alpha \sin\lambda \\ -\sin\alpha \sin\varphi \sin\lambda - \cos\alpha \cos\lambda \\ \sin\alpha \cos\varphi \end{bmatrix}.$$

We return to Clairaut's equation (1.65) and differentiate with respect to arc length $s$:

$$\frac{d(N \cos\varphi \sin\alpha)}{ds} = \frac{dN}{ds} \cos\varphi \sin\alpha - N \sin\varphi \sin\alpha \frac{d\varphi}{ds} - N \cos\varphi \cos\alpha \frac{d\alpha}{ds} = 0.$$

Inserting (1.70) and (1.63) yields

$$\frac{d\alpha}{ds} = \frac{1}{N} \tan\varphi \sin\alpha. \tag{1.72}$$

Using (1.63), (1.64), and (1.72) we get

$$\boldsymbol{n}' = \begin{bmatrix} \frac{dn_1}{d\varphi}\frac{d\varphi}{ds} + \frac{dn_1}{d\lambda}\frac{d\lambda}{ds} \\ \frac{dn_2}{d\varphi}\frac{d\varphi}{ds} + \frac{dn_2}{d\lambda}\frac{d\lambda}{ds} \\ \frac{dn_3}{d\varphi}\frac{d\varphi}{ds} + \frac{dn_3}{d\lambda}\frac{d\lambda}{ds} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} V^2 \sin\varphi \cos\lambda \cos\alpha + \sin\lambda \sin\alpha \\ V^2 \sin\varphi \sin\lambda \cos\alpha - \cos\lambda \sin\alpha \\ -V^2 \cos\varphi \cos\alpha \end{bmatrix}.$$

From (1.47) we have

$$\begin{aligned} \tau_g &= |\boldsymbol{w}_1 \quad \boldsymbol{w}_3 \quad \boldsymbol{w}'{}_3| = |\boldsymbol{t} \quad \boldsymbol{n} \quad \boldsymbol{n}'| = \boldsymbol{n}' \cdot (\boldsymbol{t} \times \boldsymbol{n}) = \boldsymbol{n}' \cdot \boldsymbol{b} \\ &= \frac{1}{N}(\sin\alpha \cos\alpha - V^2 \sin\alpha \cos\alpha) = -\frac{e' \cos^2\varphi \cos\alpha \sin\alpha}{N}. \end{aligned} \tag{1.73}$$

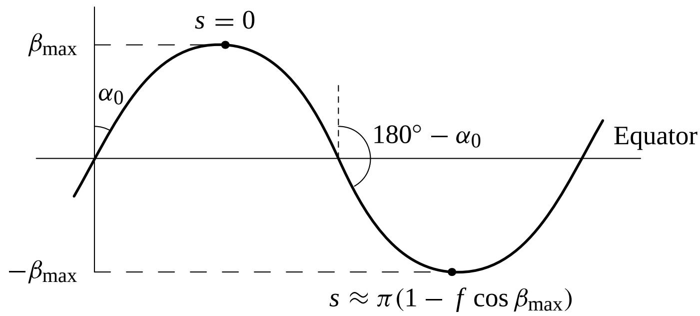


**Figure 1.8** The global behaviour of a geodesic. The arc length is counted from a point with latitude $\beta_{\max}$ and positive Eastwards. The full period of the geodesic is approximately $2\pi(1 - f\cos\beta_{\max})$

In the last step we used (1.10). From (1.19) we get

$$\kappa_n = \frac{\cos^2\alpha}{M} + \frac{\sin^2\alpha}{N} = \frac{1 + e'^2\cos^2\varphi\cos^2\alpha}{N}. \tag{1.74}$$

Recall from (1.50) that for the geodesic $\kappa_g = 0$.

Now we have found expressions for $\frac{d\varphi}{ds} = \dots$, $\frac{d\lambda}{ds} = \dots$, and $\frac{d\alpha}{ds} = \dots$ which are first order differential equations that make the starting point for series expansions for the geodesic. However we choose another procedure and expose the oldest and maybe the most efficient one published by F. W. Bessel in 1826.

Before going through Bessel's method we first make us familiar with the global behaviour of the geodesic.

## 1.7 The Behaviour of the Geodesic

For a given reduced latitude $\beta$ the product $\cos\beta\sin\alpha$ in Clairaut's equation (1.67) attains its maximal value when $\alpha = 90°$. Assume that a geodesic starts at $P_1$ on the Northern hemisphere with azimuth $\alpha_1$ ($0° < \alpha_1 < 90°$). When moving Eastwards along the geodesic the reduced latitude $\beta$ increases and $\cos\beta$ decreases. Hence we must have an increasing $\sin\alpha$ so that at the same time $\beta$ increases and $\alpha$ increases until finally $\alpha = 90°$ is achieved.

After obtaining the maximal value $\beta_{\max}$, $\beta$ decreases because the product $\cos\beta\sin\alpha$ cannot increase and cannot stay constant. In the latter case the parallel would be a geodesic and the parallel certainly is not a geodesic as its osculating plane inclines to the surface! With increasing $\cos\beta$ follows that $\sin\alpha$ decreases. Now $\alpha > 90°$ so $\alpha$ gets smaller and smaller.

From (1.67) we have

$$\sin\alpha_0 = \cos\beta_{\max}$$

or

$$\alpha_0 + \beta_{\max} = 90°. \tag{1.75}$$

After crossing the Equator $\cos\beta$ decreases and $\sin\alpha$ decreases and this continues until again $\alpha = 90°$ and $\beta$ attains its maximal negative value. Next again $\sin\alpha$ decreases and $\cos\beta$ increases until a new crossing of Equator happens and the geodesic is on the Northern hemisphere. For $\alpha_{\min}$ and $\alpha_{\max}$ we have

$$\alpha_{\min} + \alpha_{\max} = 180°. \tag{1.76}$$

The geodesic continues forward with azimuths between these extremal values, that appear when crossing the Equator.

Any geodesic runs between a Northern and a Southern parallel of equal latitude. This latitude depends on the constant $\cos\beta\sin\alpha$. For a zero constant either $\alpha = 0°$ or $\beta = 90°$. The first case is the meridian, the second case gives no line. If the constant is $\pm 1$ we get $\alpha = 90°$ or $270°$ and $\beta = 0$. This corresponds to the Equator.

## 1.8 The Direct and the Inverse Geodetic Problems

We present Bessel's method for solving the direct and inverse geodetic problems on the ellipsoid. We use Bessel's auxiliary sphere and form differential equations similar to those applied in Bessel's original method. However, they are integrated in a different way than the original one. The coefficients resulting from the multiplications of the series expansion and the power functions themselves are stored and used later in the development. The coefficients of the expansion and the powers are all transformed to three nested coefficients $K_1$, $K_2$, and $K_3$, and two ellipsoidal corrections $\Delta\sigma$ and $\Delta\omega$ including terms up to $e^8$ or $e'^8$. Therefore the accuracy of the method is higher than that of others. The following is based on Xue-Lian (1985).

Figure 1.9 shows the geodesic elements on the ellipsoid. Figure 1.10 shows the corresponding elements transformed on the unit auxiliary sphere one by one. The North Pole is designated by $\mathcal{N}$.

### 1.8.1 Basic differential equations for $ds$ and $d\sigma$, $d\lambda$ and $d\omega$

From (1.63) and (1.64) we get the differential equations of a geodesic on an ellipsoid

$$\left.\begin{aligned} ds &= \frac{M}{\cos\alpha}\, d\varphi \\ d\lambda &= \frac{M\tan\alpha}{N\cos\varphi}\, d\varphi \end{aligned}\right\}. \tag{1.77}$$

On the unit sphere this is

$$\left.\begin{aligned} d\sigma &= \frac{1}{\cos\alpha}\, d\beta \\ d\omega &= \frac{\tan\alpha}{\cos\beta}\, d\beta \end{aligned}\right\}. \tag{1.78}$$

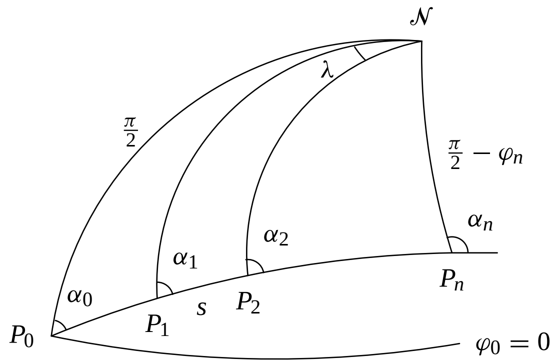


**Figure 1.9** Polar triangles on the ellipsoid. Point $P_i$ has latitude and longitude $(\varphi_i, \lambda_i)$. The length of the geodesic from $P_0$ to $P_i$ is $s_i$, especially is $s = s_2 - s_1$, azimuth of geodesic at $P_i$ is $\alpha_i$

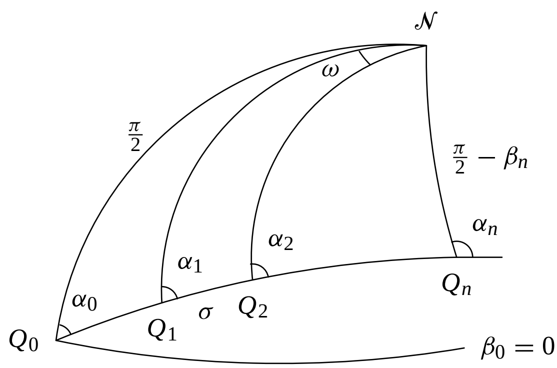


**Figure 1.10** Polar triangles on the unit sphere. Point $Q_i$ has reduced latitude and longitude $(\beta_i, \omega_i)$, the angular distance from $Q_0$ to $Q_i$ is $\sigma_i$, especially is $\sigma = \sigma_2 - \sigma_1$, azimuth of great circle at $Q_i$ is $\alpha_i$

Next we want to relate $s$ and $\sigma$ and $\lambda$ and $\omega$. From Figure 1.11 we read

$$\frac{d\sigma}{ds} = \frac{d\beta}{M\,d\varphi}$$


**Figure 1.11** Differential isometric triangles on ellipsoid and unit sphere

or

$$ds = M \frac{d\varphi}{d\beta} \, d\sigma. \tag{1.79}$$

We also read

$$\frac{M \, d\varphi}{d\beta} = \frac{N \cos\varphi \, d\lambda}{d\omega}$$

or

$$d\lambda = \frac{M \cos\beta}{N \cos\varphi} \frac{d\varphi}{d\beta} \, d\omega. \tag{1.80}$$

We differentiate (1.12): $\tan\beta = (1-f)\tan\varphi$ and remember (1.14): $\frac{\cos\varphi}{\cos\beta} = W$. From (1.14) and (1.2) we get $1 - f = \frac{W}{V}$ and finally

$$\frac{d\varphi}{d\beta} = VW. \tag{1.81}$$

Hence by (1.79) and (1.80), considering $M = \frac{b}{V} W^2$ and from (1.18): $\frac{M}{N} = \frac{1}{V^2}$, we have

$$\left.\begin{aligned} ds &= \frac{b}{W} \, d\sigma \\ d\lambda &= \frac{1}{V} \, d\omega \end{aligned}\right\}. \tag{1.82}$$

Substituting (1.14) we obtain the important equations

$$ds = b\sqrt{1 + e'^2 \sin^2\beta} \, d\sigma \tag{1.83}$$

$$d\lambda = \sqrt{1 - e^2 \cos^2\beta} \, d\omega. \tag{1.84}$$

### 1.8.2 The formulas for transformation of $\sigma$ to $s$

Using spherical trigonometric formulas for the right angled triangle $QQ_n\mathcal{N}$ in Figure 1.10, we obtain

$$\sin\beta = \sin\beta_n \sin\sigma. \tag{1.85}$$

Substituting (1.85) into (1.83), the function of $\beta$ in (1.83) is reduced to one of $\sigma$. We expand $\sqrt{1 + e'^2 \sin^2\beta}$ to include the term contining $e'^8$ and introduce the abbreviation

$$t = \tfrac{1}{4} e'^2 \sin^2\beta_n \tag{1.86}$$

and get

$$ds = b(1 + 2t\sin^2\sigma - 2t^2\sin^4\sigma + 4t^3\sin^6\sigma - 10t^4\sin^8\sigma) \, d\sigma. \tag{1.87}$$

We substitute the sine power functions of $\sigma$ in (1.87) by the cosine functions of multiple angles:

$$\left.\begin{aligned} \sin^2\sigma &= \tfrac{1}{2}(1 - \cos 2\sigma) \\ \sin^4\sigma &= \tfrac{1}{8}(3 - 4\cos 2\sigma + \cos 4\sigma) \\ \sin^6\sigma &= \tfrac{1}{32}(10 - 15\cos 2\sigma + 6\cos 4\sigma - \cos 6\sigma) \\ \sin^8\sigma &= \tfrac{1}{128}(35 - 56\cos 2\sigma + 28\cos 4\sigma - 8\cos 6\sigma + \cos 8\sigma) \end{aligned}\right\} \tag{1.88}$$

and get

$$ds = b\big((1 + t - \tfrac{3}{4}t^2 + \tfrac{5}{4}t^3 - \tfrac{175}{64}t^4) - t(1 - t + \tfrac{15}{8}t^2 - \tfrac{35}{8}t^3)\cos 2\sigma$$
$$- \tfrac{1}{4}t^2(1 - 3t + \tfrac{35}{4}t^2)\cos 4\sigma - \tfrac{1}{8}t^3(1 - 5t)\cos 6\sigma - \tfrac{5}{64}t^4 \cos 8\sigma\big)\, d\sigma. \quad (1.89)$$

Let

$$K_1 = 1 + t\big(1 - \tfrac{1}{4}t(3 - t(5 - 11t))\big) \quad (1.90)$$

$$K_2 = t\big(1 - t(2 - \tfrac{1}{8}t(37 - 94t))\big)$$
$$= \tfrac{1}{K_1}t(1 - t + \tfrac{15}{8}t^2 - \tfrac{35}{8}t^3). \quad (1.91)$$

Then

$$\tfrac{1}{4}K_2^2 + \tfrac{1}{16}t^4 = \tfrac{1}{K_1}\tfrac{1}{4}t^2(1 - 3t + \tfrac{35}{4}t^2)$$
$$\tfrac{1}{8}K_2^3 = \tfrac{1}{K_1}\tfrac{1}{8}t^3(1 - 5t).$$

By this (1.89) can be written as

$$ds = K_1 b\big(1 - K_2(\cos 2\sigma + \tfrac{1}{4}K_2(\cos 4\sigma + \tfrac{1}{2}K_2 \cos 6\sigma))$$
$$+ \tfrac{1}{64}t^4(1 - 4\cos 4\sigma - 5\cos 8\sigma)\big)\, d\sigma. \quad (1.92)$$

Now we use the integral formula

$$\int_{\sigma_1}^{\sigma_2} \cos n\sigma \, d\sigma = \tfrac{2}{n}\sin \tfrac{n}{2}(\sigma_2 - \sigma_1)\cos \tfrac{n}{2}(\sigma_2 + \sigma_1) \quad (1.93)$$

and introduce the abbreviations

$$s = s_2 - s_1, \qquad \sigma = \sigma_2 - \sigma_1, \qquad \sigma_m = \sigma_2 + \sigma_1 \quad (1.94)$$

and by integrating (1.92) we have

$$s = K_1 b\big(\sigma - K_2 \sin\sigma(\cos\sigma_m + \tfrac{1}{4}K_2(\cos\sigma\cos 2\sigma_m + \tfrac{1}{6}K_2(1 + 2\cos 2\sigma)\cos 3\sigma_m))$$
$$+ \tfrac{1}{64}t^4(\sigma \sin\sigma\cos\sigma(4\cos 2\sigma_m + 5\cos 2\sigma\cos 4\sigma_m))\big). \quad (1.95)$$

Finally we introduce the abbreviations

$$\Delta\sigma = K_2 \sin\sigma\big(\cos\sigma_m + \tfrac{1}{4}K_2(\cos\sigma\cos 2\sigma_m + \tfrac{1}{6}K_2(1 + 2\cos 2\sigma)\cos 3\sigma_m)\big) \quad (1.96)$$

$$\delta s = K_1 b\tfrac{1}{64}t^4\big(\sigma - \sin\sigma\cos\sigma(4\cos 2\sigma_m + 5\cos 2\sigma\cos 4\sigma_m)\big) \quad (1.97)$$

and get

$$s = K_1 b(\sigma - \Delta\sigma) + \delta s. \quad (1.98)$$

For the longest $s$ we always have $\delta s < 3 \cdot 10^{-3}$ mm. So we neglect $\delta s$ and solve for $\sigma$:

$$\sigma = \frac{s}{K_1 b} + \Delta\sigma. \quad (1.99)$$

$K_1$ and $K_2$ are called the *nested coefficients*, and $\Delta\sigma$ is called the *ellipsoidal reduction for the distance* and is computed by iteration. The accuracy for the geodetic length $s$ will be better than 0.01 mm when the approximate value of $\Delta\sigma$ is sufficiently small.

### 1.8.3 The formulas for transformation of $\omega$ to $\lambda$

We expand equation (1.84) into a series containing the term of $e^8$:

$$d\lambda = d\omega - \left(\tfrac{1}{2}e^2\cos^2\beta + \tfrac{1}{8}e^4\cos^4\beta + \tfrac{1}{16}e^6\cos^6\beta + \tfrac{5}{128}e^8\cos^8\beta\right)d\omega. \tag{1.100}$$

By (1.67)

$$\sin\alpha = \frac{\cos\beta_n}{\cos\beta}. \tag{1.101}$$

From (1.78) we get the differential equation of a geodesic on a sphere

$$d\omega = \frac{\sin\alpha}{\cos\beta}\,d\sigma.$$

Substituting (1.101) into above, then

$$d\omega = \frac{\cos\beta_n}{\cos^2\beta}\,d\sigma. \tag{1.102}$$

Substituting the above into the second term on the right side of (1.100)

$$d\lambda = d\omega - \cos\beta_n\left(\tfrac{1}{2}e^2 + \tfrac{1}{8}e^4\cos^2\beta + \tfrac{1}{16}e^6\cos^4\beta + \tfrac{5}{128}e^8\cos^6\beta\right)d\beta. \tag{1.103}$$

Reducing the function of $\beta$ to $\sigma$ with (1.85)

$$\left.\begin{aligned}\cos^2\beta &= 1 - \sin^2\beta_n\sin^2\sigma\\ \cos^4\beta &= 1 - 2\sin^2\beta_n\sin^2\sigma + \sin^4\beta_n\sin^4\sigma\\ \cos^6\beta &= 1 - 3\sin^2\beta_n\sin^2\sigma + 3\sin^4\beta_n\sin^4\sigma - \sin^6\beta_n\sin^6\sigma\end{aligned}\right\}. \tag{1.104}$$

Hence by (1.103)

$$\begin{aligned}d\lambda = d\omega - \cos\beta_n\Big(&(\tfrac{1}{2}e^2 + \tfrac{1}{8}e^4 + \tfrac{1}{16}e^6 + \tfrac{5}{128}e^8) - (\tfrac{1}{8}e^4 + \tfrac{1}{8}e^6 + \tfrac{15}{128}e^8)\sin^2\beta_n\sin^2\sigma\\ &+ (\tfrac{1}{16}e^6 + \tfrac{15}{128}e^8)\sin^4\beta_n\sin^4\sigma - \tfrac{5}{128}e^8\sin^6\beta_n\sin^6\sigma\Big)d\sigma.\end{aligned} \tag{1.105}$$

Using the expression of $f$ by $e^2$

$$f = \tfrac{1}{2}e^2 + \tfrac{1}{8}e^4 + \tfrac{1}{16}e^6 + \tfrac{5}{128}e^8 \tag{1.106}$$

as a substitute for the series of $e^2$ in (1.105) we have

$$\begin{aligned}\tfrac{1}{8}e^4 + \tfrac{1}{8}e^6 + \tfrac{5}{128}e^8 &= \tfrac{1}{2}f^2(1 + f + f^2)\\ \tfrac{1}{16}e^6 + \tfrac{15}{128}e^8 &= \tfrac{1}{8}f^3(4 + 9f)\\ \tfrac{5}{128}e^8 &= \tfrac{5}{8}f^4.\end{aligned}$$

Let

$$v = \tfrac{1}{4}f\sin^2\beta_n. \tag{1.107}$$

Then (1.105) becomes

$$d\lambda = d\omega - f\cos\beta_n\big(1 - 2v(1+f+f^2)\sin^2\sigma + 2v^2(4+9f)\sin^4\sigma - 40v^3\sin^6\sigma\big)\,d\sigma. \tag{1.108}$$

Transforming the sine power functions to the cosine functions of multiple angle with (1.88) we may write

$$\begin{aligned} d\lambda = d\omega - f\cos\beta_n\big(&1 - v(1+f+f^2 - v(3+\tfrac{27}{4}f - \tfrac{25}{2}v))\\ &+ v(1+f+f^2 - v(4+9f-\tfrac{75}{4}v))\cos 2\sigma + v^2(1+\tfrac{9}{4}f - \tfrac{15}{2}v)\cos 4\sigma\\ &+ \tfrac{5}{4}v^3\cos 6\sigma\big)\,d\sigma. \end{aligned} \tag{1.109}$$

Let

$$\lambda = \lambda_2 - \lambda_1, \qquad \omega = \omega_2 - \omega_1. \tag{1.110}$$

Using (1.93) and considering (1.94) and (1.110), equation (1.109) is integrated to be the form

$$\begin{aligned} \lambda = \omega - f\cos\beta_n\big(&(1 - v(1+f+f^2 - v(3+\tfrac{27}{4}f - \tfrac{25}{2}v)))\sigma\\ &+ v(1+f+f^2 - v(4+9f-\tfrac{75}{4}v))\sin\sigma\cos\sigma_m\\ &+ v^2(1+\tfrac{9}{4}f-\tfrac{15}{2}v)\sin\sigma\cos\sigma\cos 2\sigma\\ &+ \tfrac{5}{12}v^3\sin\sigma(1+2\cos 2\sigma))\cos 3\sigma_m\big). \end{aligned} \tag{1.111}$$

Let

$$K_3 = v\big(1+f+f^2 - v(3+7f-13v)\big). \tag{1.112}$$

Changing

$$\begin{aligned} (1-K_3) - (\tfrac{1}{4}f - \tfrac{1}{2}v)v^2 &= 1 - v\big(1+f+f^2 - v(3+\tfrac{27}{4}f - \tfrac{25}{4}v)\big)\\ (1-K_3)K_3 - \tfrac{1}{4}v^3 &= v\big(1+f+f^2 - v(4+9f-\tfrac{75}{4}v)\big)\\ (1-K_3)K_3^3 - (\tfrac{1}{4}f - \tfrac{1}{2}v)v^2 &= v^2(1+\tfrac{9}{4}f - \tfrac{15}{2}v). \end{aligned}$$

Substituting into (1.111) then

$$\begin{aligned} \lambda = \omega - (1-K_3)f\cos\beta_n&\big(\sigma + K_3\sin\sigma(\cos\sigma_m + K_3\cos\sigma\cos 2\sigma_m)\big)\\ &- \tfrac{1}{8}v^2 f^2\cos\beta_n\big((1+\cos^2 u_n)(\sigma - \cos\sigma\cos 2\sigma_m)\\ &+ \tfrac{1}{6}\sin^2\beta_n(3\sin\sigma\cos\sigma_m - 5(1+2\cos 2\sigma)\cos 3\sigma_m)\big). \end{aligned} \tag{1.113}$$

Let

$$\Delta\omega = (1-K_3)f\cos\beta_n\big(\sigma + K_3\sin\sigma(\cos\sigma_m + K_3\cos\sigma\cos 2\sigma_m)\big) \tag{1.114}$$

$$\begin{aligned} \delta\lambda = \tfrac{1}{8}v^2 f^2\cos\beta_n&\big((1+\cos^2 u_n)(\sigma - \cos\sigma\cos 2\sigma_m)\\ &+ \tfrac{1}{6}\sin^2\beta_n(3\sin\sigma\cos\sigma_m - 5(1+2\cos 2\sigma)\cos 3\sigma_m)\big). \end{aligned} \tag{1.115}$$

Then

$$\lambda = \omega - \Delta\omega - \delta\lambda. \tag{1.116}$$

The estimated maximum of $\delta\lambda$ for the largest $\lambda$ is always less than $10^{-6}$ arcsec. Similarly for $\delta S$ so it can be omitted. Hence $\lambda = \omega - \Delta\omega$ or

$$\omega = \lambda + \Delta\omega. \tag{1.117}$$

$K_3$ is called a nested coefficient too, and $\Delta\omega$ is called the *ellipsoid reduction for the difference of longitude*. It is also only found by iteration like $\Delta\sigma$. The accuracy for the longitude difference will be better than $10^{-5}$ arcsec when the change of the approximate value of $\Delta\omega$ is sufficiently small.

The nested method is based on equations (1.99) and (1.117).

### 1.8.4 Algorithms

#### Direct model

**1** $\tan\beta_1 = (1 - f)\tan\varphi_1$

**2** $\tan\sigma_1 = \dfrac{\tan\beta_1}{\cos\alpha_1}$

**3** $\cos\beta_n = \cos\beta_1 \sin\alpha_1$

**4** (1) $t = \frac{1}{4}e'^2 \sin^2\beta_n$

(2) $K_1 = 1 + t\bigl(1 - \frac{1}{4}t(3 - t(5 - 11t))\bigr)$

(3) $K_2 = t\bigl(1 - t(2 - \frac{1}{8}t(37 - 94t))\bigr)$

**5** (1) $v = \frac{1}{4} f \sin^2\beta_n$

(2) $K_3 = v\bigl(1 + f + f^2 - v(3 + 7f - 13v)\bigr)$

$\Delta\sigma = 0$

**6** (1) $\sigma = \frac{s}{K_1 b} + \Delta\sigma$

(2) $\sigma_m = 2\sigma_1 + \sigma$

**7** $\Delta\sigma = K_2 \sin\sigma\bigl(\cos\sigma_m + \frac{1}{4}K_2(\cos\sigma\cos 2\sigma_m + \frac{1}{6}K_2(1 + 2\cos 2\sigma)\cos 3\sigma_m)\bigr)$

Steps **6** and **7** are iterated until the change in $\Delta\sigma$ becomes less than a specified threshold value. The initial approximation of $\Delta\sigma$ is zero.

**8** (1) $\tan\beta_2 = \dfrac{\sin\beta_1\cos\sigma + \cos\beta_1\sin\sigma\cos\alpha_1}{\sqrt{1 - \sin^2\beta_n\sin^2(\sigma_1 + \sigma)}}$

(2) $\tan\varphi_2 = \dfrac{\tan\beta_2}{1 - f}$

**9** $\Delta\omega = (1 - K_3) f \cos\beta_n\bigl(\sigma + K_3\sin\sigma(\cos\sigma_m + K_3\cos\sigma\cos 2\sigma_m)\bigr)$

**10** (1) $\tan\omega = \dfrac{\sin\sigma\sin\alpha_1}{\cos\beta_1\cos\sigma - \sin\beta_1\sin\sigma\cos\alpha_1}$

(2) $\lambda_2 = \lambda_1 + \omega - \Delta\omega$

**11** $\tan\alpha_2 = \dfrac{\cos\beta_1 \sin\alpha_1}{\cos\beta_1 \cos\sigma \cos\alpha_1 - \sin\beta_1 \sin\sigma}$

**Inverse model**

**1** (1) $\tan\beta_1 = (1 - f)\tan\varphi_1$

(2) $\tan\beta_2 = (1 - f)\tan\varphi_2$

$\Delta\omega = 0$

**2** $\omega = \lambda_2 - \lambda_1 + \Delta\omega$

**3** $\tan\sigma = \dfrac{\sqrt{(\cos\beta_2 \sin\omega)^2 + (\cos\beta_1 \sin\beta_2 - \sin\beta_1 \cos\beta_2 \cos\omega)^2}}{\sin\beta_1 \sin\beta_2 + \cos\beta_1 \cos\beta_2 \cos\omega}$

**4** $\cos\beta_n = \dfrac{\cos\beta_1 \cos\beta_2 \sin\omega}{\sin\sigma}$

**5** $\cos\sigma_m = \cos\sigma - \dfrac{2\sin\beta_1 \sin\beta_2}{\sin^2\beta_n}$

**6** (1) $v = \frac{1}{4} f \sin^2\beta_n$

(2) $K_3 = v\big(1 + f + f^2 - v(3 + 7f - 13v)\big)$

**7** $\Delta\omega = (1 - K_3) f \cos\beta_n\big(\sigma + K_3 \sin\sigma(\cos\sigma_m + K_3 \cos\sigma \cos 2\sigma_m)\big)$

The procedure is iterated starting with step **2** and ending with step **7** repeatedly until the change in $\Delta\omega$ is negligible compared to a specified threshold value. The initial approximation of $\Delta\omega$ is zero.

**8** (1) $t = \frac{1}{4} e'^2 \sin^2\beta_n$

(2) $K_1 = 1 + t\big(1 - \frac{1}{4}t(3 - t(5 - 11t))\big)$

(3) $K_2 = t\big(1 - t(2 - \frac{1}{8}t(37 - 94t))\big)$

**9** $\Delta\sigma = K_2 \sin\sigma\big(\cos\sigma_m + \frac{1}{4}K_2(\cos\sigma \cos 2\sigma_m + \frac{1}{6}K_2(1 + 2\cos 2\sigma)\cos 3\sigma_m)\big)$

**10** $s = K_1 b(\sigma - \Delta\sigma)$

**11** (1) $\tan\alpha_1 = \dfrac{\cos\beta_2 \sin\omega}{\cos\beta_1 \sin\beta_2 - \sin\beta_1 \cos\beta_2 \cos\omega}$

(2) $\tan\alpha_2 = \dfrac{\cos\beta_1 \sin\omega}{\cos\beta_1 \sin\beta_2 \cos\omega - \sin\beta_1 \cos\beta_2}$

***Note 1*** *The thresholds for the changes in $\Delta\sigma$ or $\Delta\omega$ both can be chosen as $10^{-12}$, $10^{-10}$, or $10^{-8}$. Then the corresponding errors of $\varphi$, $\lambda$, and $s$ are shown in Table 1.2 as $d\varphi$, $d\lambda$, and $ds$ respectively.*

These algorithms are coded as the *M*-files bessel_d and bessel_i.

**Table 1.2** Thresholds and Errors

| Threshold | $d\varphi$ or $d\lambda$ [arcsec] | $ds$ [mm] |
|---|---|---|
| $10^{-12}$ | $10^{-5}$ | 1 |
| $10^{-10}$ | $10^{-4}$ | 3 |
| $10^{-8}$ | $10^{-3}$ | 5 |

## 1.9 The Ellipsoidal System Extended to Outer Space

To determine a point $P$ above the surface of the Earth, we use geographical coordinates $\varphi$, $\lambda$ and the *height* $h$. The height above the ellipsoid is measured along a perpendicular line. Let a point $Q$ on the ellipsoid have coordinates $\varphi$, $\lambda$. In a geocentric $X, Y, Z$-system, with $X$-axis at longitude $\lambda = 0$, the point $Q$ with height $h = 0$ has coordinates

$$X = N \cos\varphi \cos\lambda \tag{1.118}$$

$$Y = N \cos\varphi \sin\lambda \tag{1.119}$$

$$Z = (1 - f)^2 N \sin\varphi. \tag{1.120}$$

The distance to the $Z$-axis along the normal at $Q$ is $N = a \big/ \sqrt{1 - f(2 - f)\sin^2\varphi}$. This is the radius of curvature in the direction perpendicular to the meridian (the prime vertical).

The formula for $N$ results from substitution of $p = N\cos\varphi$ and (1.2) into (1.1). When $P$ is above the ellipsoid, we must add to $Q$ a vector of length $h$ along the normal. From spatial geographical $(\varphi, \lambda, h)$ we get spatial Cartesian $(X, Y, Z)$:

$$X = (N + h) \cos\varphi \cos\lambda \tag{1.121}$$

$$Y = (N + h) \cos\varphi \sin\lambda \tag{1.122}$$

$$Z = \big((1 - f)^2 N + h\big) \sin\varphi. \tag{1.123}$$

**Example 1.3** Let $P$ be given by the coordinates $(\varphi, \lambda, h)$ in the WGS 84 system:

$$\varphi = 57^\circ\, 01'\, 45.464\,54'' = 57.029\,295\,69^\circ,$$
$$\lambda = \ \ 9^\circ\, 57'\, 00.893\,21'' = \ \ 9.950\,248\,114^\circ \text{ and}$$
$$h = 56.950\,\text{m}$$

We seek the $(X, Y, Z)$ coordinates of $P$. The result is achieved by the $M$-file geo2cart:

$$X = 3\,426\,949.397\,\text{m}$$
$$Y = \ \ 601\,195.852\,\text{m}$$
$$Z = 5\,327\,723.994\,\text{m}.$$

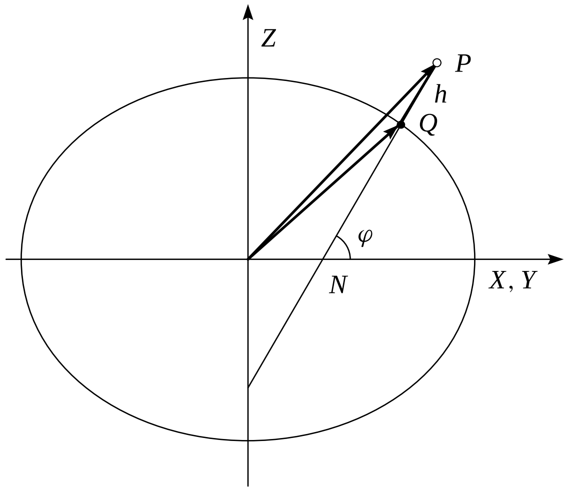


**Figure 1.12** Conversion between $(\varphi, \lambda, h)$ and Cartesian $(X, Y, Z)$

The reverse problem—compute $(\varphi, \lambda, h)$ from $(X, Y, Z)$—requires an iteration for $\varphi$ and $h$. Directly $\lambda = \arctan(Y/X)$. There is quick convergence for $h \ll N$, starting at $h = 0$:

$$\varphi \text{ from } h \text{ (1.123):} \qquad \varphi = \arctan\left(\frac{Z}{\sqrt{X^2+Y^2}}\left(1 - \frac{(2-f)fN}{N+h}\right)^{-1}\right) \tag{1.124}$$

$$h \text{ from } \varphi \text{ (1.121)–(1.122):} \qquad h = \frac{\sqrt{X^2+Y^2}}{\cos\varphi} - N. \tag{1.125}$$

For large $h$ (or $\varphi$ close to $\pi/2$) we recommend the procedure given in the $M$-file c2gm.

**Example 1.4** Given the same point as in Example 1.3, the $M$-file cart2geo solves the reverse problem. The result agrees at mm-level with the original values for $(\varphi, \lambda, h)$ in Example 1.3.

Using GPS for positioning we have to convert coordinates from Cartesian $(X, Y, Z)$ to spatial geographical $(\varphi, \lambda, h)$ and vice versa as described in (1.118)–(1.120) or (1.124)–(1.125). The immediate result of a satellite positioning is a set of $X, Y, Z$-values which we most often want to convert to $\varphi, \lambda, h$-values.

# 2

# Conformal Mappings of the Ellipsoid

ABSTRACT All mappings used in geodesy are conformal. This implies that dicrctions only need a second order correction while distances need a noteable correction that cannot be neglected.

We establish the necessary theory for understanding the characteristics of the mappings. We derive the explicit formulas for the transformations $(\varphi, \lambda) \to (x, y)$ and $(x, y) \to (\varphi, \lambda)$ as well as formulas for the meridian convergence and the scale.

These formulas are adopted for the Universal Transverse Mercator mapping (UTM).

Finally we add comments on the issue of defining the best possible mapping.

## 2.1 Conformality

Globally the Universal Transverse Mercator mapping (UTM) is by far the most used one. To a lesser extent conformal conical and stereographic mappings are in use.

For UTM the variation of scale is of the order of $4 \cdot 10^{-4}$. Conformal mappings have small the corrections for directions. The image of an infinitesimal figure fits—conforms to—the original figure in the sense that they have approximately the same form. However $m$ varies from point to point, so a large figure may have an image the form of which deviates markedly from the original figure. This fact is very desirable as small figures in the field hence become similar to the mapped ones. Any conformal mapping can be conceived as a generalisation of a similarity transformation.

The scale $m$ is a scalar function of position which is described by the two parameters $(x, y)$. In case $m$ is constant the mapping becomes a proper similarity transformation. If especially $m = 1$ the mapping is isometric.

UTM is based on a conformal mapping theory described by C. F. Gauss and we want to give a detailed exposition of just this method. In doing so we follow Großmann (1976).

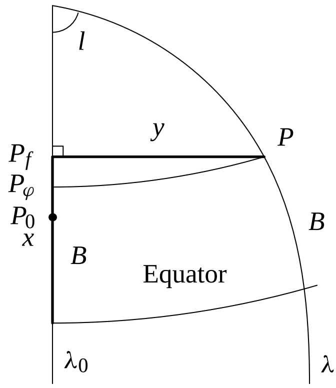


**Figure 2.1** The Cartesian 2-D plane with pertinent mappings of selected ellipsoidal quantities

We use a 2-D Cartesian coordinate system with *$x$-axis positive to the North and $y$-axis positive to the East*, confer Figure 2.1. In this chapter we use the unit of degrees as this is still the prevailing unit for geographical positions.

Finally we want to stress that we consiously avoid the term projection. Often one sees simple drawings illustrating how ellipsoidal points are mapped onto the plane. These drawings more often introduce misunderstandings than bring clarity to the subject. We warn against having any confidence in their message. We are of the opinion that mappings shall be described purely analytically. So that we do and at the same time try to avoid using Taylor's formula more often than necessary.

## 2.2 The Gaussian Mapping of the Ellipsoid onto the Plane

The problem is to map conformally a geographical coordinate system $(\varphi, \lambda)$ onto a plane Cartesian coordinate system $(x, y)$ under the following conditions:

**1** The central meridian having $\lambda_0 =$ constant must be mapped onto the $x$-axis of the plane system.

**2** The plane image of the central meridian shall be mapped distance true.

We present the conformal mapping which C. F. Gauss used in the years 1820–30 for computation of the Hanoverian grade measurement. Gauss himself computed more than 3000 point in his new mapping. We shall develop formulas for transformation from $(\varphi, \lambda)$ to $(x, y)$ and formulas for the *scale, meridian convergence*, and *corrections of distance and direction*.

The concept of conformality is intimately connected to analytical functions. If $x$ and $y$ are real variables then $z = x + iy$ is said to be a complex variable. Here $i$ denotes the *imaginary unit*, $i^2 = -1$. $x$ is called the real part of $z$ and $y$ is the imaginary part of $z$. A function $f(z)$ is called analytical at a point $z = z_0$ if $f$ is defined and has a derivative at any point in a neighbourhood of $z_0$. The function is called analytical in a region $\mathcal{D}$ if it is analytical at any point in $\mathcal{D}$. Polynomials with complex constants $c_0, \dots, c_n$ of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n \tag{2.1}$$

are analytical in the whole plane.

The length differential $dS$ can be expressed in geographical coordinates $(\varphi, \lambda)$:

$$dS^2 = M^2\, d\varphi^2 + (N \cos\varphi)^2\, d\lambda^2.$$

We want to use complex function theory. This is only possible if $d\varphi$ and $d\lambda$ have a factor in common. To obtain this, we introduce the following substitution:

$$dq = \frac{M}{N \cos\varphi}\, d\varphi. \tag{2.2}$$

The new variable $q$ is called ***isometric latitude***. Let $e$ denote the eccentricity, then the explicit expression for $q$ is

$$q = \int_0^{\varphi} \frac{M}{N \cos\xi}\, d\xi = \ln\Bigg(\tan\big(\tfrac{\pi}{4} + \tfrac{\varphi}{2}\big)\Big(\frac{1 - e\sin\varphi}{1 + e\sin\varphi}\Big)^{e/2}\Bigg). \tag{2.3}$$

Substituting $dq$ we get

$$dS^2 = N^2 \cos^2\varphi (dq^2 + d\lambda^2).$$

Now let $l$ be the difference in longitude between the meridian containing the arbitrary point $P$ and the central meridian $\lambda_0$. We describe a conformal mapping through the analytical function $F$:

$$x + iy = F(q + il). \tag{2.4}$$

The origin at the ellipsoid of revolution is taken as the intersection between the central meridian and Equator. This gives the first boundary condition ($l = 0$ for $y = 0$), condition **1** above:

$$x_{y=0} = F(q).$$

The second boundary condition tells that the meridian arc from Equator to the point with isometric latitude $q$ (corresponding to the geographical latitude $\varphi$) equals $B$, condition **2** above:

$$x_{y=0} = B = F(q). \tag{2.5}$$

Consequently the analytical function $F$ measuress the meridional arc as a function of isometric latitude. We are not interested in $F$ itself, but rather its derivatives:

$$dx = dB = Md\varphi = N \cos\varphi\, dq.$$

The last equality follows from (2.2). We divide by $q$:

$$\frac{dx}{dq} = \frac{dB}{dq} = \frac{dF(q)}{dq} = F'(q) = N \cos\varphi.$$

According to the assumptions $l = \lambda - \lambda_0$ is small (a few degrees), but $\varphi$ and $q$ can take on any value between 0° og 90°. In order to work with small values both for $l$ and $q$ we introduce an auxillary point $P_0$ on the central meridian. (In case of Denmark $P_0$ often has the value of 56° north latitude.) In all further computations we work with ***differences in geographic and isometric latitude***:

$$\varphi - \varphi_0 = \Delta\varphi, \qquad q - q_0 = \Delta q.$$

We insert these differences into (2.4)

$$x + iy = F(q_0 + \Delta q + il).$$

Next we expand $F$ into a Taylor series around the point $P_0$:

$$x + iy = F(q)_0 + F'(q)_0(\Delta q + il) + \frac{1}{2}F''(q)_0(\Delta q + il)^2 + \frac{1}{6}F'''(q)_0(\Delta q + il)^3 + \cdots . \tag{2.6}$$

Remember from (2.5) that $F(q)_0 = B$ so

$$x + iy = B + \left(\frac{dB}{dq}\right)_0 (\Delta q + il) + \frac{1}{2}\left(\frac{d^2B}{dq^2}\right)_0 (\Delta q + il)^2 + \frac{1}{6}\left(\frac{d^3B}{dq^3}\right)_0 (\Delta q + il)^3 + \cdots . \tag{2.7}$$

Let the coefficients in the series be $a_1, a_2, \ldots$ and we put $x - B = \Delta x$. Then the approximated mapping equation is

$$\Delta x + iy = a_1(\Delta q + il) + a_2(\Delta q + il)^2 + a_3(\Delta q + il)^3 + \cdots \tag{2.8}$$

with

$$\left.\begin{aligned}
a_1 &= \left(\frac{dB}{dq}\right)_0 = \left(\frac{dB}{d\varphi}\frac{d\varphi}{dq}\right)_0 = \left(M\frac{N\cos\varphi}{M}\right)_0 = N_0\cos\varphi_0\\
a_2 &= \frac{1}{2}\left(\frac{d^2B}{dq^2}\right)_0 = \frac{1}{2}\left(\frac{d}{d\varphi}\frac{dB}{dq}\frac{d\varphi}{dq}\right)_0 = -\frac{1}{2}N_0\cos^2\varphi_0\, t_0\\
a_3 &= \frac{1}{6}\left(\frac{d^3B}{dq^3}\right)_0 = \frac{1}{6}\left(\frac{d}{d\varphi}\frac{d^2B}{dq^2}\frac{d\varphi}{dq}\right)_0 = -\frac{1}{6}N_0\cos^3\varphi_0(1 - t_0^2 + \eta_0^2)\\
a_4 &= \frac{1}{24}\left(\frac{d^4B}{dq^4}\right)_0 = \frac{1}{24}\left(\frac{d}{d\varphi}\frac{d^3B}{dq^3}\frac{d\varphi}{dq}\right)_0 = \frac{1}{24}N_0\cos^4\varphi_0\, t_0(5 - t_0^2 + 9\eta_0^2 + 4\eta_0^4)\\
a_5 &= \frac{1}{120}\left(\frac{d^5B}{dq^5}\right)_0 = \frac{1}{120}\left(\frac{d}{d\varphi}\frac{d^4B}{dq^4}\frac{d\varphi}{dq}\right)_0 = \frac{1}{120}N_0\cos^5\varphi_0(5 - 18t_0^2 + t_0^4 + \cdots)
\end{aligned}\right\} . \tag{2.9}$$

Separating the real and the imaginary parts in (2.8) we obtain the series for $\Delta x$ and $\Delta y$. On the right side we have powers of $\Delta q$ and $l$.

Still looking for simple formulas we introduce a trick: In stead of using the auxiliary point $P_0$ on the central meridian as expansion point we introduce another point of expansion, namely $P_\varphi$ which also is on the central meridian and having the same latitude $\varphi$ as the point to be mapped. Then follows that $\Delta\varphi = \Delta q = 0$ and in the parenteses in (2.8) is only left $il$. Separation into the real and the imaginary parts yields

$$\left.\begin{aligned} x &= B - a_2 l^2 + a_4 l^4 + \cdots \\ y &= a_1 l - a_3 l^3 + a_5 l^5 - \cdots \end{aligned}\right\}, \tag{2.10}$$

The coefficients $a_i$ shall be computed for latitude $\varphi$. The final mapping equations read

$$\left.\begin{aligned} x &= B + \frac{1}{2} N \cos^2\varphi\, t l^2 + \frac{1}{24} N \cos^4\varphi\, t (5 - t^2 + 9\eta^2) l^4 + \cdots \\ y &= N \cos\varphi\, l + \frac{1}{6} N \cos^3\varphi (1 - t^2 + \eta^2) l^3 \\ &\qquad + \frac{1}{120} N \cos^5\varphi (5 - 18t^2 + t^4) l^5 + \cdots \end{aligned}\right\}. \tag{2.11}$$

## 2.3 Transformation of Cartesian to Geographical Coordinates

The transformation consists of two steps. First we compute the isometric ellipsoidal coordinates from the Cartesian coordinates. Next the isometric latitude is converted to geographical latitude. We change from Cartesian to isometric coordinates by using the inverse function $f$ of $F$:

$$q + il = f(x + iy). \tag{2.12}$$

The boundary conditions for $F$ are

$$q_{l=0} = f(x) = f(B) \quad \text{and} \quad f'(x) = \frac{dq}{dB} = \frac{1}{N \cos\varphi}.$$

The right side of (2.12) can again be developed at the point $P_0$ and with $q - q_0 = \Delta q$ and $x - B_0 = \Delta x$, we get

$$\begin{aligned} \Delta q + il &= \left(\frac{dq}{dB}\right)_0 (\Delta x + iy) + \frac{1}{2}\left(\frac{d^2q}{dB^2}\right)_0 (\Delta x + iy)^2 \\ &\qquad + \frac{1}{6}\left(\frac{d^3q}{dB^3}\right)_0 (\Delta x + iy)^3 + \cdots \end{aligned} \tag{2.13}$$

or introducing a set of coefficients $b_i$:

$$\begin{aligned} \Delta q + il &= b_1(\Delta x + iy) + b_2(\Delta x + iy)^2 + b_3(\Delta x + iy)^3 \\ &\qquad + b_4(\Delta x + iy)^4 + b_5(\Delta x + iy)^5 + \cdots . \end{aligned} \tag{2.14}$$

Explicitly the coefficients $b_i$ are

$$\left.\begin{aligned} b_1 &= \left(\frac{dq}{dB}\right)_0 = \frac{1}{N_0\cos\varphi_0} \\ b_2 &= \frac{1}{2}\left(\frac{d^2q}{dB^2}\right)_0 = \frac{t_0}{2N_0^2\cos\varphi_0} \\ b_3 &= \frac{1}{6}\left(\frac{d^3q}{dB^3}\right)_0 = \frac{1+2t_0^2+\eta_0^2}{6N_0^3\cos\varphi_0} \\ b_4 &= \frac{1}{24}\left(\frac{d^4q}{dB^4}\right)_0 = \frac{t_0(5+6t_0^2+\eta_0^2-4\eta_0^4)}{24N_0^4\cos\varphi_0} \\ b_5 &= \frac{1}{120}\left(\frac{d^5q}{dB^5}\right)_0 = \frac{5+28t_0^2+24t_0^4+\cdots}{120N_0^5\cos\varphi_0} \end{aligned}\right\}. \tag{2.15}$$

Again the series expansion takes place at the footpoint $P_f = (\varphi_f, 0)$ of $P_0$. So $\Delta x = 0$ and the separation into the real and the imaginary parts yields

$$\left.\begin{aligned} \Delta q &= -b_2y^2 + b_4y^4 - \cdots \\ l &= b_1y - b_3y^3 + b_5y^5 - \cdots \end{aligned}\right\}. \tag{2.16}$$

The coefficients $b_i$ must be computed for $\varphi_f$.

We do not seek $\Delta q$ but rather $\Delta\varphi$. The transition from $\Delta q$ to $\Delta\varphi$ also happens by means of a series in which we introduce $\Delta\varphi = \varphi - \varphi_f$ and $\Delta q = q - q_f$

$$\varphi - \varphi_f = d_1\Delta q + d_2(\Delta q)^2 + \cdots$$

with

$$d_1 = \left(\frac{d\varphi}{dq}\right)_f = \cos\varphi_f(1+\eta_f^2); \quad d_2 = \frac{1}{2}\left(\frac{d^2\varphi}{dq^2}\right)_f = -\frac{1}{2}\cos^2\varphi_f\, t_f(1+4\eta_f^2).$$

Insertion of $\Delta q$ yields

$$\varphi - \varphi_f = -b_2d_1y^2 + (b_4d_1 + b_2^2d_2)y^4 + \cdots.$$

Finally, by inserting the coefficients $b_i$ and $d_i$ we get

$$\left.\begin{aligned} \varphi &= \varphi_f - \frac{t_f}{2N_f^2}(1+\eta_f^2)y^2 + \frac{t_f}{24N_f^4}(5+3t_f^2+6\eta_f^2-6\eta_f^2t_f^2)y^4 + \cdots \\ l &= \frac{1}{N_f\cos\varphi_f}y - \frac{1+2t_f^2+\eta_f^2}{6N_f^3\cos\varphi_f}y^3 + \frac{5+28t_f^2+24t_f^4}{120N_f^5\cos\varphi_f}y^5 + \cdots \end{aligned}\right\}. \tag{2.17}$$

For $|l| < 3°$ these series guarantee an accuracy at the mm-level when the latitude $\varphi$ is not too large because then $t$ increases rapidly.

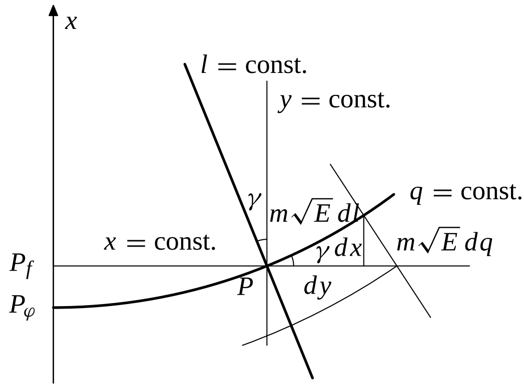


**Figure 2.2** Meridian convergence

## 2.4 Meridian Convergence and Scale

The meridian convergence $\gamma$ at a point $P$ is the angle between the image of the meridian $l =$ constant and the coordinate line $y =$ constant. $\gamma$ is reckogned clockwise from the northern branch of $l =$ constant. The same angle appears between the plane curve $q =$ constant and $x =$ constant, confer Figure 2.2.

Differentially we have

$$\tan\gamma = \frac{dx}{dy},$$

or for $\varphi =$ constant

$$\tan\gamma = \frac{\dfrac{dx}{dl}}{\dfrac{dy}{dl}}.$$

From (2.10) we have

$$\tan\gamma = \frac{-2a_2 l + 4a_4 l^3 + \cdots}{a_1 - 3a_3 l^2 + \cdots} = \frac{-2a_2 l\left(1 - \frac{2a_4}{a_2} l^2 + \cdots\right)}{a_1\left(1 - \frac{3a_3}{a_1} l^2 + \cdots\right)}$$
$$= -\frac{2a_2 l}{a_1}\left[1 + \left(-\frac{2a_4}{a_2} + \frac{3a_3}{a_1}\right) l^2 + \cdots\right].$$

We insert (2.9) and omit index

$$\tan\gamma = \sin\varphi\, l + \frac{1}{3}\sin\varphi\cos^2\varphi(1 + t^2 + 3\eta^2) l^3 + \cdots.$$

We use the series $\gamma = \tan\gamma - \frac{1}{3}\tan^3\gamma + \cdots$ and get the *plane Gaussian meridian convergence* in unit of radian

$$\gamma = \sin\varphi\, l + \frac{1}{3}\sin\varphi\cos^2\varphi(1 + 3\eta^2)\, l^3 + \cdots. \qquad (2.18)$$

From Figure 2.2 we also get for $x =$ constant:

$$\tan\gamma = \frac{m\sqrt{E}\,dq}{m\sqrt{E}\,dl} = \frac{\frac{\partial q}{\partial y}}{\frac{\partial l}{\partial y}}$$

An expansion similar to above yields

$$\tan\gamma = \frac{-2b_2y + 4b_4y^3 + \cdots}{b_1 - 3b_3y^3 + \cdots} = \frac{2b_2y}{b_1}\left[1 + \left(-\frac{2b_4}{b_2} + \frac{3b_3}{b_1}\right)y^2 + \cdots\right]$$

eller med koefficienterne fra (2.15) og $\tan\gamma = \gamma + \cdots$

$$\gamma = \frac{t_f}{N_f}y - \frac{t_f}{3N_f^3}(1 + t_f^2 - \eta_f^2)y^3 + \cdots .$$

For conformal mapping there exists a close relationship between scale $m$ and meridian convergence $\gamma$. However, this connection only can be demonstrated to its full extent and beauty by using analytical functions. Here we limit ourselves to quote the following result:

$$m\sin\gamma = -\frac{\partial y}{\partial B}; \quad m\cos\gamma = \frac{\partial x}{\partial B}.$$

These equations can be checked by insertion into (2.19) below.

***Scale*** $m$ ***as function of*** $\varphi$ ***and*** $l$ The *scale* $m$ is defined at the ratio between the elementary arc element on the ellipsoid $dS$ and its plane image $ds$ in the Cartesian coordinate system:

$$m^2 = \frac{ds^2}{dS^2} = \frac{E}{E}\frac{dx^2 + dy^2}{dq^2 + dl^2} = \frac{1}{N^2\cos^2\varphi}\left[\left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2\right]$$

$$= \frac{1}{N^2\cos^2\varphi}\left[\left(\frac{\partial x}{\partial B}\right)^2 + \left(\frac{\partial y}{\partial B}\right)^2\right]\left(\frac{dB}{dq}\right)^2.$$

Remember that $dB/dq = N\cos\varphi$ and consequently

$$m^2 = \left(\frac{\partial x}{\partial B}\right)^2 + \left(\frac{\partial y}{\partial B}\right)^2. \tag{2.19}$$

From (2.9) and (2.10) we get

$$\frac{\partial x}{\partial B} = 1 - \frac{da_2}{dq}\frac{dq}{dB}l^2 + \frac{da_4}{dq}\frac{dq}{dB}l^4 + \cdots$$

$$\frac{\partial y}{\partial B} = \frac{da_1}{dq}\frac{dq}{dB}l - \frac{da_3}{dq}\frac{dq}{dB}l^3 + \cdots ,$$

$$\frac{da_1}{dq} = 2a_2, \quad \frac{da_2}{dq} = 3a_3, \quad \frac{da_3}{dq} = 4a_4, \quad \frac{da_4}{dq} = 5a_5, \quad \frac{dq}{dB} = a_1^{-1},$$

$$\frac{\partial x}{\partial B} = 1 + \frac{1}{2}\cos^2\varphi(1 - t^2 + \eta^2)l^2 + \frac{1}{24}\cos^4\varphi(5 - 18t^2 + t^4)l^4 + \cdots$$
$$\frac{\partial y}{\partial B} = -\cos\varphi\, t\, l - \frac{1}{6}\cos^3\varphi\, t(5 - t^2 + \cdots)l^3 + \cdots .$$

Insertion into (2.19) gives

$$m^2 = 1 + \cos^2\varphi(1 + \eta^2)l^2 + \frac{1}{3}\cos^4\varphi(2 - t^2 + \cdots)l^4 + \cdots$$

or taking the square root

$$m = 1 + \frac{1}{2}\cos^2\varphi(1 + \eta^2)l^2 + \frac{1}{24}\cos^4\varphi(5 - 4t^2 + \cdots)l^4 + \cdots .$$

***Scale*** $m$ ***as function of*** $y$ A similar expression for $m$ as function of $y$ can be detrmned through the following derivation. We start from the inverse expression of (2.19):

$$\frac{1}{m^2} = \left(\frac{\partial B}{\partial x}\right)^2 + \left(\frac{\partial B}{\partial y}\right)^2 .$$

So we have to calculate the partial derivatives of $B$ with respect to $x$ and $y$. Therefore we must find expressions for $B$ that are functions of those variables. This can be achieved by expanding $f(B) = f\big(x + (B - x)\big)$ into a Taylor series:

$$f(B) = f(x) + f'(x)(B - x) + \tfrac{1}{2} f''(x)(B - x)^2 + \cdots . \tag{2.20}$$

By insertion of

$$f(B) = q, \qquad f(x) = q_f, \qquad f'(x) = \left(\frac{dq}{dB}\right)_f, \text{ etc.}$$

and (2.15), equation (2.20) becomes:

$$q - q_f = \Delta q = b_1(B - x) + b_2(B - x)^2 + \cdots .$$

But from (2.16) we also have

$$\Delta q = -b_2 y^2 + b_4 y^4 + \cdots .$$

Equating the two expressions and solving for $(B - x)$ we get

$$B - x = \frac{-b_2}{b_1}y^2 + \frac{b_4}{b_1}y^4 - \frac{b_2}{b_1}(B - x)^2 + \cdots .$$

We approximate $(B - x)$ by the first term on the right side:

$$B = x - \frac{b_2}{b_1}y^2 + \left(\frac{b_4}{b_1} - \frac{b_2^3}{b_1^3}\right)y^4 + \cdots .$$

From (2.15) we insert $b_i$

$$B = x - \frac{t_f}{2N_f}y^2 + \frac{t_f}{24N_f^3}(5 + 3t_f^2 + \cdots)y^4 + \cdots . \tag{2.21}$$

Now we differentiate at the point $P_f$

$$\frac{\partial B}{\partial x} = -\frac{t_f}{N_f}y + \frac{1}{6N_f^3}(5t_f + 3t_f^3 + \cdots)y^3 + \cdots .$$

When we want to derive with respect to $x$ we explicitly have to go via geographical latitude $\varphi$:

$$\frac{\partial B}{\partial x} = \frac{\partial B}{\partial \varphi}\frac{\partial \varphi}{\partial x} = \frac{\partial B}{\partial \varphi}\frac{\partial \varphi}{\partial B} = \frac{\partial B}{\partial \varphi}\frac{V^2}{N}.$$

We differentiate (2.21) with respect to $x$ and use

$$\frac{dt}{d\varphi} = 1 + t^2, \qquad \frac{dN^{-1}}{d\varphi} = -\frac{\eta^2 t}{NV^2}$$

$$\frac{\partial B}{\partial x} = 1 - \frac{1}{2N_f^2}(1 + t_f^2 + \eta_f^2)y^2 + \frac{1}{24N_f^4}(5 + 14t_f^2 + 9t_f^4 + \cdots)y^4 + \cdots .$$

Insertion into (2.4) yields

$$\frac{1}{m^2} = 1 - \frac{1 + \eta_f^2}{N_f^2}y^2 + \frac{2}{3N_f^4}y^4 + \cdots$$

or finally from the binomial formula

$$m = 1 + \frac{1 + \eta_f^2}{2N_f^2}y^2 + \frac{1}{24N_f^4}y^4 + \cdots .$$

As $(1 + \eta^2)/N = 1/M$ this expression becomes in *spherical approximation*

$$m = 1 + \frac{1}{2R^2}y^2 + \frac{1}{24R^4}y^4 + \cdots .$$

## 2.5 Multiplication of the Mapping

On page 30 under **2** we asked that the central meridian should be mapped one-to-one (distance true). This condition can generalize so that a distance at the central meridian is mapped into a distance on the $x$-axis so that the two distances make a constant proportion $m_0 \neq 1$. By this (2.5) changes to

$$F(q) = m_0 B. \tag{2.22}$$

Repeating earlier derivations we find that $x$ and $y$ in (2.11) and (2.17) likewise shall be multiplied by $m_0$. If we designate coordinates defined by (2.22) with $\bar{x}$ and $\bar{y}$ we have

$$\bar{x} = m_0 x \qquad \text{and} \qquad \bar{y} = m_0 y.$$

Choosing an appropriate value for $m_0$ we can keep the scale within certain limits. By choosing $m_0 < 1$ we may suppress a too large value for the scale at distances far from the central meridian. A guideline may be to fix $m_0$ to a value at the central meridian that is exactly as much less than unity as the scale becomes larger than unity at the boundary of the mapping area. However, a reasonable "round value" is to be preferred.

## 2.6 Universal Transverse Mercator

Based on the Gaussian mapping the Department of Defence in U.S.A. introduced shortly after the Second World War the Universal Transverse Mercator Grid System (UTM). The reference ellipsoid is the International Ellipsoid of 1924. The mapping is defined for the whole Earth. It was a desire to limit the variation of the scale $m$ so the total mapping of the Earth is divided into 60 sections that are named zones. Each zone cover 6° in longitude and they are numbered from 1 to 60. Number 1 covers 180°–174° West, number 2 covers 174°–168° West, and so on. The scale at the central meridian is $m_0 = 0.999\,6$. The UTM system is limited by the parallels 84° North and 80° South. The polar regions are mapped stereographically.

Abscissas $x$ are called *northings* $N$, and ordinates $y$ are called *eastings* $E$. Each zone has its own coordinate system. The central meridian has a false easting ($FE$) of 500 000 m and the Equator has a false northing ($FN$) of 0 m for points on the northern hemisphere and a false northing of 10 000 000 m for points on the southern hemisphere. This simple arrangement leaves all coordinates positive, and creates a total mess in the equatorial area.

The transformation of geographical coordinates $(\varphi, \lambda)$ into UTM coordinates $(N, E)$ and revesely appear so often in practice that we in detail in Examples 2.1 and 2.2 below demonstrate how to solve the problems on a pocket calculator. We also point to the MATLAB codes geo2utm and utm2geo:

```
[N, E] = geo2utm(phi, lambda, zone)
[phi, lambda] = utm2geo(N, E, zone)
```

**Example 2.1** Transformation of geographical coordinates to UTM coordinates.

Given (9° is the longitude for the central meridian in zone 32):

$$\begin{aligned}\varphi &= 57^\circ\,01'\,45.464\,5'' = 57.029\,295\,69^\circ \text{ North}\\ l &= \lambda - \lambda_0 = 9^\circ\,57'\,00.893\,2'' - 9^\circ = 0.950\,248\,111^\circ\\ &= 0.016\,584\,958\,\text{rad, East}\end{aligned}$$

Search: $(N, E)$

We start by computing the meridional arc $B$ from Equator to latitude $\varphi$ according to (1.28):

$$\begin{aligned}B &= 6\,338\,038.411 - 14\,738.870\,10(1 - 0.001\,252\,443 + 0.000\,001\,965)\\ &= 6\,323\,317.972\,\text{m}\end{aligned}$$

The further computations run according to (2.11):

$$
\begin{aligned}
t &= 1.541\,590\,030 \\
\eta^2 &= e'^2 \cos^2\varphi = 0.006\,768\,170 \cos^2\varphi = 0.002\,004\,493 \\
N &= \frac{c}{V} = \frac{c}{\sqrt{1+\eta^2}} = 6\,393\,531.919\,\text{m} \\
x &= B + \frac{N}{2}\cos^2\varphi\, t l^2 + \frac{N}{24}\cos^4\varphi\, t(5 - t^2 + 9\eta^2) l^4 \\
&= 6\,323\,317.972 + 401.459 + 0.007 = 6\,323\,719.439\,\text{m} \\
y &= N\cos\varphi\left[1 + \frac{1}{6}\cos^2\varphi(1 - t^2 + \eta^2)l^2\right. \\
&\qquad\qquad\qquad \left. + \frac{1}{120}\cos^4\varphi(5 - 18t^2 + t^4)l^4\right] l \\
&= 57\,706.116\,48(1 - 0.000\,018\,662 - 0.000\,000\,002) \\
&\qquad\qquad\qquad\qquad\qquad = 57\,705.039\,\text{m}.
\end{aligned}
$$

Finally we get in zone 32

$$
\begin{aligned}
N &= FN + m_0 x = 0 + 0.999\,6 \cdot 6\,323\,719.439 = 6\,321\,189.95\,\text{m} \\
E &= FE + m_0 y = 500\,000 + 0.999\,6 \cdot 57\,705.039 = 557\,681.96\,\text{m}
\end{aligned}
$$

**Example 2.2** Transformation of UTM coordinates to geographical coordinates.

Given: $(N, E)$ = (6 321 189.95, 557 681.96) in zone 32.

Search: $(\varphi, \lambda)$

The first step is to compute the latitude $\varphi_f$ for the footpoint, see Figure 1.1. The computation occurs iteratively according to (1.28). The first approximation is

$$
\varphi_f^{(1)} = \frac{\frac{N}{m_0}}{0.994\,955\,869c}.
$$

With the approximate value $\varphi_f^{(1)}$ we compute $B^{(1)}$ from (1.28) and the next approximation is

$$
\varphi_f^{(2)} = \frac{\frac{N}{m_0} - B^{(1)}}{0.994\,955\,869c}.
$$

The procedure is as follows

$$
\varphi_f^{(i+1)} = \varphi_f^{(i)} + \frac{\frac{N}{m_0} - B^{(i)}}{0.994\,955\,869c}
$$

and the result

$$
\varphi_f = \varphi_f^{(n)}, \quad \text{when} \quad \frac{N}{m_0} - B^{(n)} = 0.
$$

Note that any $B^{(i)}$ must be computed directly from (1.28). The computation now runs as follows

$$\varphi_f^{(1)} = \frac{6\,323\,719.438}{6\,367\,654.496} = 0.993\,100\,276$$
$$B^{(1)} = 6\,308\,969.718\,\text{m}$$
$$\varphi_f^{(2)} = 0.993\,100\,276 + \frac{6\,323\,719.438 - 6\,308\,969.718}{6\,367\,654.496} = 0.995\,416\,627$$
$$B^{(2)} = 6\,323\,749.608\,\text{m}$$
$$\varphi_f^{(3)} = 0.995\,416\,627 + \frac{6\,323\,719.438 - 6\,323\,749.608}{6\,367\,654.496} = 0.995\,411\,889$$
$$B^{(4)} = 6\,323\,719.378\,\text{m}$$
$$\varphi_f^{(4)} = 0.995\,411\,889 + \frac{6\,323\,719.438 - 6\,323\,719.378}{6\,367\,654.496} = 0.995\,411\,898$$
$$B^{(4)} = 6\,323\,719.438\,\text{m}$$

So $N/m_0 - B^{(4)} \approx 0$ and hence $\varphi_f = 0.995\,411\,898\,\text{rad}$. The subsequent computations are

$$t_f = 1.541\,802\,493, \qquad \eta_f^2 = 0.002\,004\,104$$
$$N_f = \frac{c}{\sqrt{1+\eta_f^2}} = 6\,393\,533.165\,\text{m}$$
$$y = \frac{E - FE}{m_0} = \frac{557\,681.96 - 500\,000}{0.999\,6} = 57\,705.040\,\text{m}$$

We insert into (2.17) and get

$$\varphi = 0.995\,411\,898 - 0.000\,062\,924 + 0.000\,000\,005 = 0.995\,348\,979\,\text{rad}$$
$$l = 0.016\,586\,254(1 - 0.000\,078\,152 + 0.000\,000\,011) = 0.016\,584\,958\,\text{rad}$$
$$\lambda = 9^\circ + l.$$

Finally

$$\varphi = 57^\circ\,01'\,45.464\,4''$$
$$\lambda = \ \ 9^\circ\,57'\,00.893\,1''$$

which result is in agreement with the starting values in Example 2.1—except for the influence of rounding errors.

## 2.7 The Gaussian Conformal Mapping of the Sphere into the Plane

In practical surveying we must know how measured directions and distances shall be corrected when mapped conformally from the ellipsoid into the plane. As mentioned earlier

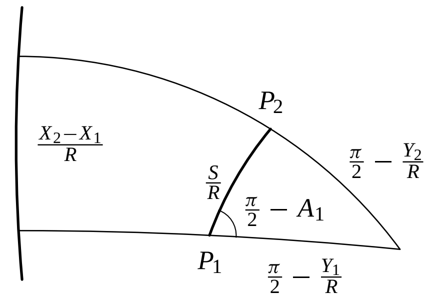


**Figure 2.3** Transfer of spherical coordinates. All lines are great circles

the distance correction is so large that is has to be considered in most cases. The direction corretion, however, often is small because of conformality. For both types of observation we only are interested in finding *correction* terms, so we substitute the ellipsoid with a Gaussian sphere with radius $R = \sqrt{M_0 N_0}$. Here $M_0$ and $N_0$ are computed as mean values for the area. The expressions derived are valid for side lengths up to 40 km.

Our present goal could be reached by setting $\eta = 0$ in (2.11) and (2.17). However for pedagogical reasons we want to repeat all derivations, but now in a spherical approximation based on spherical trigonometry.

Referring to Figur 2.3 we assume that a point $P_1$ with coordinates $(X_1,Y_1)$ as well as the distance $S$ and the bearing $A_1$ are given. We seek $(X_2, Y_2)$. In fact we are dealing with a polar computation on the sphere.

We compute $Y_2$ from the spherical cosine law

$$\cos\left(\frac{\pi}{2} - \frac{Y_2}{R}\right) = \cos\frac{S}{R}\cos\left(\frac{\pi}{2} - \frac{Y_1}{R}\right) + \sin\frac{S}{R}\sin\left(\frac{\pi}{2} - \frac{Y_1}{R}\right)\cos\left(\frac{\pi}{2} - A_1\right)$$

or

$$\sin\frac{Y_2}{R} = \cos\frac{S}{R}\sin\frac{Y_1}{R} + \sin\frac{S}{R}\cos\frac{Y_1}{R}\sin A_1.$$

For small angles we expand the sine and cosine functions including two terms and we get

$$\frac{Y_2}{R} - \frac{Y_2^3}{6R^3} = \left(1 - \frac{S^2}{2R^2}\right)\left(\frac{Y_1}{R} - \frac{Y_1^3}{6R^3}\right) + \left(\frac{S}{R} - \frac{S^3}{6R^3}\right)\left(1 - \frac{Y_1^2}{2R^2}\right)\sin A_1 + \cdots,$$

or rearranged

$$Y_2 = Y_1\left(1 - \frac{S^2}{2R^2} - \frac{Y_1^2}{6R^2}\right) + S\sin A_1\left(1 - \frac{S^2}{6R^2} - \frac{Y_1^2}{2R^2}\right) + \frac{Y_2^3}{6R^2} + \cdots. \quad (2.23)$$

The last right side term is rewritten in the following manner

$$\frac{Y_2^3}{6R^2} = \frac{\left(Y_1 + S\sin A_1\right)^3}{6R^2}$$
$$= \frac{Y_1^3}{6R^2} + \frac{Y_1^2 S\sin A_1}{2R^2} + \frac{Y_1 S^2 \sin^2 A_1}{2R^2} + \frac{S^3\sin^3 A_1}{6R^2} + \cdots$$

and is inserted into (2.23)

$$Y_2 = Y_1 + S\sin A_1 - \frac{Y_1 S^2\cos^2 A_1}{2R^2} - \frac{S^3 \sin A_1 \cos^2 A_1}{6R^2} - \cdots.$$

With the new variables

$$S\cos A_1 = \Delta X \quad \text{and} \quad S\sin A_1 = \Delta Y, \tag{2.24}$$

the ordinate formula becomes

$$Y_2 = Y_1 + S\sin A_1 - \frac{Y_1(\Delta X)^2}{2R^2} - \frac{(\Delta X)^2\Delta Y}{6R^2} - \cdots. \tag{2.25}$$

For *computation of* $X_2$ we use the spherical sine law

$$\sin\frac{X_2 - X_1}{R} = \sin\frac{S}{R}\,\frac{\sin\left(\frac{\pi}{2} - A_1\right)}{\sin\left(\frac{\pi}{2} - Y_2\right)} = \frac{\sin\frac{S}{R}}{\cos\frac{Y_2}{R}}\cos A_1.$$

All small angles are substituted by their series expansions

$$\frac{X_2 - X_1}{R} - \frac{(X_2 - X_1)^3}{6R^3} = \left(\frac{S}{R} - \frac{S^3}{6R^3}\right)\left(1 + \frac{Y_2^2}{2R^2}\right)\cos A_1 + \cdots \tag{2.26}$$

$$X_2 - X_1 = S\cos A_1\left(1 - \frac{S^2}{6R^2} + \frac{Y_2^2}{2R^2}\right) + \frac{(X_2 - X_1)^3}{6R^2} + \cdots. \tag{2.27}$$

We get $(X_2 - X_1)^3/6R^2 = S^3\cos^3 A_1/6R^2 + \cdots$. By this the formula for the transfer of the abscissa is

$$X_2 = X_1 + S\cos A_1 + \frac{Y_2^2 S\cos A_1}{2R^2} - \frac{S^3\sin^2 A_1\cos A_1}{6R^2} + \cdots$$

or by using (2.24)

$$X_2 = X_1 + S\cos A_1 + \frac{Y_2^2\Delta X}{2R^2} - \frac{\Delta X(\Delta Y)^2}{6R^2} + \cdots. \tag{2.28}$$

If you want to work with explicit formulas $Y_2$ can be substituted by $Y_1 + \Delta Y + \cdots$. The formulas (2.25) and (2.28) for $Y_2$ and $X_2$ are analogue to the plane formulas for computation of a traverse; now two correction terms are added; they vanish in the plane case ($R = \infty$).

By this we have derived explicit series for the computation of the coordinates for a new point based on the coordinates of a given point and a direction leaving from this given point and with a known distance. In geodesy this setup also is named *the direct problem*.

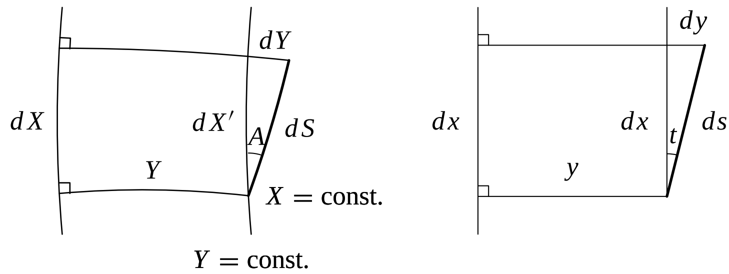


**Figure 2.4** Infinitesimal spherical triangle and its conformal image

In this section up till now we have dealt with a simple case of coordinate computation on the sphere. Next we define the *Gaussian conformal mapping* of the sphere into the plane.

We start from a differential spherical right angled triangle whose small sides are parallel to the meridian and the parallel. The two smaller sides must be mapped proportionally. The differential arc $dX'$ of the spherical parallel $Y =$ constant is mapped as

$$dx = \frac{1}{\cos\left(\frac{Y}{R}\right)} dX' = dX. \tag{2.29}$$

The image of $dY$ which is an arc of the spherical ordinate circle $X =$ constant then must be

$$dy = \frac{1}{\cos\left(\frac{Y}{R}\right)} dY = \left(1 + \frac{Y^2}{2R^2} + \frac{5Y^4}{24R^4} + \cdots\right) dY. \tag{2.30}$$

We integrate (2.29) and (2.30). With the conditions $X = 0$ when $x = 0$ and $Y = 0$ when $y = 0$, we get the mapping equations

$$x = X \tag{2.31}$$

$$y = Y + \frac{Y^3}{6R^2} + \frac{Y^5}{24R^4} + \cdots, \tag{2.32}$$

or after reversing the series

$$X = x, \qquad Y = y - \frac{y^3}{6R^2} - \frac{y^5}{24R^4} - \cdots.$$

The scale $m$ is determined by

$$m^2 = \frac{ds^2}{dS^2} = \frac{dx^2 + dy^2}{dX'^2 + dY^2} = \frac{dx^2 + dy^2}{\cos^2\left(\frac{Y}{R}\right) dx^2 + \cos^2\left(\frac{Y}{R}\right) dy^2} = \frac{1}{\cos^2\left(\frac{Y}{R}\right)}$$

or

$$m = 1 + \frac{Y^2}{2R^2} + \frac{5Y^4}{24R^4} + \cdots \tag{2.33}$$

or after introducing $y$

$$m = 1 + \frac{y^2}{2R^2} + \frac{y^4}{24R^4} + \cdots. \tag{2.34}$$

It is characteristic for the Gaussian mapping that scale is independent of the direction angle $t$ of the arc element. On the contrary, if the scale for a mapping is independent of the direction angle of the arc element then the mapping is conformal. This has nothing to with the fact that even a conformal mapping has a direction correction, especially for large side lenghts. This is the object of our investigations in the next section.

But first we want to demonstrate another connection by transforming (2.30):

$$d\frac{y}{R} = \frac{d\frac{Y}{R}}{\cos\left(\frac{Y}{R}\right)} = \frac{1}{\tan\left(\frac{Y}{2R}+\frac{\pi}{4}\right)}\frac{d\frac{Y}{R}}{2\cos^2\left(\frac{Y}{2R}+\frac{\pi}{4}\right)}.$$

After the substitution

$$u = \tan\left(\frac{Y}{2R}+\frac{\pi}{4}\right), \qquad du = \frac{d\frac{Y}{R}}{2\cos^2\left(\frac{Y}{2R}+\frac{\pi}{4}\right)}$$

follows after integration

$$\frac{y}{R} = \int \frac{d\frac{Y}{R}}{\cos\frac{Y}{R}} = \int \frac{du}{u} = \ln u = \ln\tan\left(\frac{Y}{2R}+\frac{\pi}{4}\right). \tag{2.35}$$

As $Y = 0$ for $y = 0$, the integration constant vanishes. If we imagine a rotation of $\pi/2$ of the system, the central meridian becomes the Equator and $Y/R$ the geographical latitude. Then the right side of (2.35) becomes equal to the Mercator function: The Gaussian conformal mapping of the sphere can be persived as a transversal Mercator mapping.

## 2.8 Arc-to-Chord Correction

With reference to Figure 2.5 we aim at determining the angle $T_1 - t_1$. This quantity is called the *arc-to-chord correction*. Figure 2.5 is not to be looked upon as an infinitesimal figure, like for example Figure 2.4. The spheroidal points $P_1$ and $P_2$ are mapped into the plane points $P_1'$ and $P_2'$. In Figure 2.5 $s'$ denotes the plane image of $S$ while $s$ is the straight line between $P_1'P_2'$. The direction angle for $T$ is $s'$ and the direction angle for $t$ is $s$. The mapping is conformal so the direction angle $T$ for the image curve $s'$ equals the direction angle $A$ for the great circle $S$. We rewrite (2.25) and (2.28) in order to determine $\tan T_1$:

$$S\sin T_1 = \Delta Y + \frac{(\Delta X)^2}{6R^2}(3Y_1+\Delta Y) = \Delta Y\left(1+\frac{\Delta X}{6R^2}\frac{\Delta X}{\Delta Y}(2Y_1+Y_2)\right)$$
$$S\cos T_1 = \Delta X - \frac{\Delta X}{6R^2}\left(3Y_2^2-(\Delta Y)^2\right) = \Delta X\left(1-\frac{-Y_1^2+2Y_2^2+2Y_1Y_2}{6R^2}\right).$$

We make the ratio between the leftmost and rightmost expressions and use the binomial formula:

$$\tan T_1 = \frac{\Delta Y}{\Delta X}\left[1+\frac{\Delta X}{6R^2}\frac{\Delta X}{\Delta Y}(2Y_1+Y_2)+\frac{1}{6R^2}(-Y_1^2+2Y_2^2+2Y_1Y_2)\right]. \tag{2.36}$$

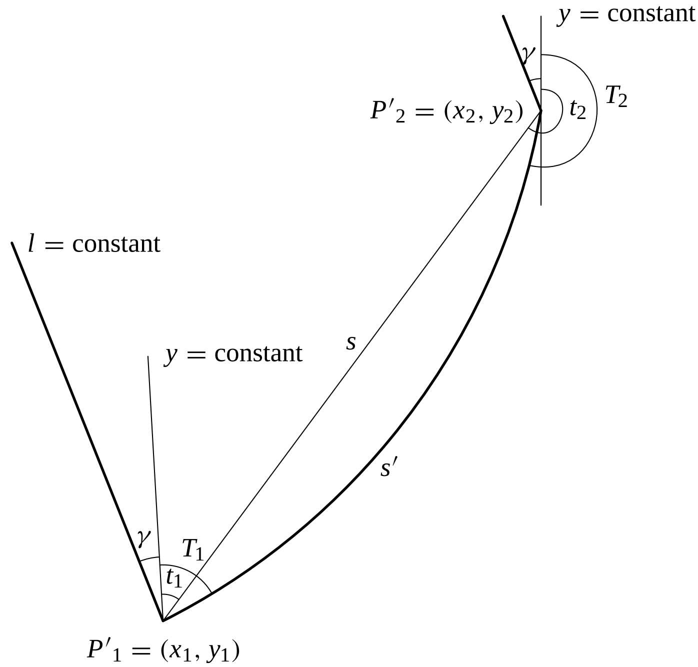


**Figure 2.5** Spherical distance conformally mapped into the plane

Without loss of accuracy we can in correction terms containing $R^{-2}$ substitute spheroidal coordinates $(X, Y)$ with plane coordinates $(x, y)$, confer also (2.31) and (2.32)

$$\tan T_1 = \frac{Y_2 - Y_1}{X_2 - X_1} + \frac{x_2 - x_1}{6R^2}(2y_1 + y_2) + \frac{1}{6R^2}(-y_1^2 + 2y_1y_2 + 2y_2^2)\tan t_1.$$

According to (2.31) (2.32) the direction angle $t_1$ for the straight line $P_1'$ between $P_2'$ is given as

$$\tan t_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{Y_2 - Y_1}{X_2 - X_1} + \frac{1}{x_2 - x_1}\frac{(y_2^3 - y_1^3)}{6R^2}$$

or

$$\tan t_1 = \frac{Y_2 - Y_1}{X_2 - X_1} + \frac{1}{6R^2}\frac{y_2 - y_1}{x_2 - x_1}(y_1^2 + y_1y_2 + y_2^2). \tag{2.37}$$

Subtracting (2.36) and (2.37) and using $\tan t_1 = (y_2 - y_1)/(x_2 - x_1)$ yields:

$$\tan T_1 - \tan t_1 = \frac{\Delta x}{6R^2}(2y_1 + y_2) + \frac{1}{6R^2}(-2y_1 + y_1y_2 + y_2^2)\tan t_1. \tag{2.38}$$

We have $1 = \cos^{-2} t - \tan^2 t = \cos^{-2} t - (\Delta y/\Delta x)\tan t$ so the first term on the right side of (2.38) can be rewritten as follows

$$\frac{\Delta x}{6R^2}(2y_1 + y_2) = \frac{\Delta x(2y_1 + y_2)}{6R^2\cos^2 t_1} - \frac{1}{6R^2}(-2y_1^2 + y_1y_2 + y_2^2)\tan t_1.$$

We insert this into (2.38) and the last terms vanish and we get

$$\tan T_1 - \tan t_1 = \frac{\Delta x(2y_1 + y_2)}{6R^2 \cos^2 t_1}.$$

A Taylor expansion yields

$$\tan t_1 = \tan\bigl(T_1 + (t_1 - T_1)\bigr) = \tan T_1 + \frac{t_1 - T_1}{\cos^2 T_1} + \frac{\sin T_1}{\cos^3 T_1}(t_1 - T_1)^2 + \cdots$$

or

$$\tan T_1 - \tan t_1 = \frac{T_1 - t_1}{\cos^2 T_1}.$$

By this the *arc-to-chord correction* at the spheroidal point $P_1$ for the Gaussian mapping of the sphere into the plane

$$T_1 - t_1 = \frac{(x_2 - x_1)(2y_1 + y_2)}{6R^2}. \tag{2.39}$$

We emphasize that in formula (2.39) $(x_2 - x_1)$ denotes a difference of coordinates in north-south direction and $y$ is the ordinate in east-west direction relative to the central meridian, confer Figure 2.5.

**Example 2.3** Let $x_2 - x_1 = 10\,\text{km}$, $y_1 = 95\,\text{km}$, $y_2 = 105\,\text{km}$ then $y_{\frac{1}{3}} = 98.3\,\text{km}$ and we obtain $T_1 - t_1 = 0.77\,\text{mgon}$. The direction correction at $P_2$ is given as $T_2 - t_2 = (x_1 - x_2)y_{\frac{1}{3}}/2R^2$ with $y_{\frac{1}{3}} = 101.6\,\text{km}$. The numerical result is $-0.79\,\text{mgon}$.

## 2.9 Correction of Distance

The correction of distance is easy to determine by integration of the expression for the scale. By using the binomial formula for (2.34) the result becomes

$$dS = \left(1 - \frac{y^2}{2R^2}\right) ds + \cdots.$$

Ignoring fourth order terms we can set $ds = dy/\sin t$ and get by integration

$$\begin{aligned} S &= s - \int_{y_1}^{y_2} \frac{y^2}{2R^2} \frac{dy}{\sin t} = s - \frac{y_2^3 - y_1^3}{6R^2 \sin t} = s - \frac{1}{6R^2} \frac{y_2 - y_1}{\sin t} \frac{y_2^3 - y_1^3}{y_2 - y_1} \\ &= s - \frac{s}{6R^2}(y_1^2 + y_1 y_2 + y_2^2). \end{aligned}$$

Accordingly the *correction of distance* for the Gaussian mapping of the sphere onto the plane for the distance between points $P_1$ and $P_2$ is

$$\sigma = S - s = -\frac{s}{6R^2}(y_1^2 + y_1 y_2 + y_2^2). \tag{2.40}$$

## 2.10 Summary of Corrections

We study the mapped meridian and the mapped geodesic through the point $P_i$. The mapped azimuth $\alpha_m$ between the tangent to the mapped meridian and the tangent to the mapped geodesic is identical to the ellipsoidal azimuth $\alpha_e$ since the mapping is conformal.

The grid azimuth $T$ of the mapped geodesic is the angle between grid north $x_m$ and the tangent to the mapped geodesic. The grid azimuth of the chord $t$ is the angle between grid north and the chord $P_i P_j$.

To keep computations simple we utilize chords and quantites related to them as follows:

– An ellipsoidal azimuth is reduced to the grid azimuth by first subtracting the meridian convergence

$$T_{ij} = \alpha_{e,ij} - \gamma_i .$$

The grid azimuth is further reduced by the arc-to-chord correction to give the grid azimuth of the chord

$$t_{ij} = T_{ij} - (T - t)_{ij} .$$

– An ellipsodal direction $d_e$ is reduced to the plane direction $d_m$ by the same direction

$$d_{m,ij} = d_{e,ij} - (T - t)_{ij} .$$

– An ellipsoidal angle $a_{e,ijk}$ between dirctions $ij$ and $ik$ is reduced by

$$a_{m,ijk} = a_{e,ijk} + (T - t)_{ij} - (T - t)_{ik} .$$

– A length of the geodesic on the ellipsoid $s_e$ is transformed to the length of the mapped geodesic by the line scale $k^*$ and we get

$$l_{ij} = s_{m,ij} = k^* s_{e,ij}$$

where the line scale $k^*$ is the average point scale over the line.

The expressions for $\gamma$, $T - t$, and $k$ vary from one conformal mapping to another. The reader is referred to Lee (1976).

## 2.11 *Best Possible* Conformal Mapping

If one were to design a new mapping with a given purpose one might ask if certain values for the scale are to be preferred to others. We then must stress that it is no quality in itself that the scale deviates less possible from unity. And we already have seen an example where we simply change the scale by multiplying by a constant, see Section 2.5.

The basic problem was investigated by Milnor (1969). He wanted to find the best possible conformal mapping of a given area that must be simply connected and have a twice differentiable boundary. "Best possible" is to be understood in the sense that the ratio $m_{\text{max}}/m_{\text{min}}$ is minimum. The problem has one solution that is derived form a boundary curve having constant scale. This result is due to Chebyshev and has been known for more than hundred years.

Throughout this chapter our starting point always has been how the mapping from the ellipsoid to the plane is defined. The description has been given in form of the explicit mapping function $F$ or as necessary geometrical descriptions of the mapping. From this we have derived—someone will maybe call it tediously—expressions for the mapping of coordinates, meridian convergence, scale, etc.

However, based on a lemma by Milnor we could have started in the following way: "We consider ***the*** conformal mapping whose *scale* is given as ...."

MILNOR'S LEMMA: *The sacle $m$ for a conformal mapping $F$ of the elliposoid determine $F$ up to a rotation and two translations in the plane. A positive function $m$ given on the ellipsoid is connected by a conformal mapping function $F$ if and only if $m$ is twice differentiable and satisfy the differential equation*

$$m^2 \Delta \ln m = \text{Gaussian curvature of the ellipsoid} = \frac{1}{MN}.$$

In 2-D the Laplace operator $\Delta$ is $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. To demonstrate a use of the lemma by checking if the UTM mapping really is conformal

$$m = c_0 + c_2 y^2 + c_4 y^4 + \cdots .$$

In case $m$ only depends on $y$ then $\Delta$ shall be interpreted as $\Delta = \frac{d^2}{dy^2}$. As $m$ varies symmetrically around the central meridian the power series only can contain even powers. We get

$$\frac{d \ln m}{dy} = \frac{1}{m} \frac{dm}{dy}$$

and

$$\frac{d^2 \ln m}{dy^2} = \left( m \frac{d^2 m}{dy^2} - \left( \frac{dm}{dy} \right)^2 \right) \Big/ m^2$$

or according to Milnor

$$m^2 \frac{d^2 \ln m}{dy^2} = m \frac{d^2 m}{dy^2} - \left( \frac{dm}{dy} \right)^2 = \frac{1}{MN}.$$

We have

$$\frac{dm}{dy} = \frac{y}{R^2} + \frac{4y^3}{24R^4} + \cdots$$

$$\frac{d^2 m}{dy^2} = \frac{1}{R} + \frac{12y^2}{23R^4} + \cdots$$

and inserting up to and including the fourth power we get

$$\begin{aligned}\frac{1}{MN} &= (c_0 + c_2 y^2 + c_4 y^4)(2c_2 + 12c_4 y^2) - (2c_2 y + 4c_4 y^3)^2 \\ &= 2c_0 c_2 + (12c_0 c_4 - 2c_2^2) y^2 - 2c_2 c_4 y^4 + \cdots .\end{aligned}$$

For $y = 0$ we get $c_2 = 1/2c_0 MN$. The coefficient to $y^2$ must be zero as the expression for $MN$ is independent of $y$; hence $c_4 = c_2^2/6c_0$.

$$m = c_0 + \frac{1}{2c_0 MN} y^2 + \frac{1}{24 c_0^3 M^2 N^2} y^4 + \cdots , \tag{2.41}$$

which in spherical approximation is identical to (2.34) for $c_0 = 1$.

Mapping freaks may find much pleasure in reading König & Weise (1951).

# References

Bodem ller, H. (1954). *Die geodätischen Linien des Rotationsellipsoides und die Lösung der geodätischen Hauptaufgaben für große Strecken unter besonderer Berücksichtigung der Bessel-Helmertschen Lösungsmethode*, volume 13 of *Deutsche Geodätische Kommission, Reihe B.* Deutsches Geodätisches Forschungsinstitut, Abteilung I, München.

Gauss, Carl Frierich (1827). *Disquisitiones generales circa superficies curvas*. A. Wangerin, Leipzig.

Großmann, Walter (1976). *Geodätische Rechnungen und Abbildungen in der Landesvermessung*. Konrad Wittwer, Stuttgart, 3rd edition.

Guggenheimer, Heinrich W. (1977). *Differential Geometry*. Dover Publications, New York.

K nig, R. & Weise, K. (1951). *Mathematische Grundlagen der höheren Geodäsie und Kartographie*. Springer, Berlin Göttingen Heidelberg.

Lee, L. P. (1976). *Conformal Projections Based on Elliptic Functions*, volume 16 of *Cartographica Monograph*. B. V. Gutsell, Toronto, Canada.

Lehn, C. J. (1957). *The Main Problem of Geodesy and the Inverse Problem Solved by Vector Methods*, volume XXVI of *Geodætisk Instituts Skrifter, 3. række*. Geodætisk Institut, København.

Milnor, John (1969). A problem in cartography. *The American Mathematical Monthly*, 76:1101–1112.

Struik, Dirk J. (1988). *Lectures on Classical Differential Geometry*. Dover Publications, New York, second edition.

Willmore, T. J. (1959). *An Introduction to Differential Geometry*. Clarendon Press, Oxford.

Xue-Lian, Zhan (1985). The nested coefficient method for accurate solutions of direct and reverse geodetic problems with any length. In *7th International Symposium on Geodetic Computations, Cracow, June 18–21*, pages 747–763, Institute of Geodesy and Cartography. Warsow, Poland, ul. Jasna 2/4.

# Index