Working with Torben Krarup

For the first time I heard from a colleague of the Danish Geodetic Institute about a mathematical genius at this Institute whom, as he said, I should meet by all means. His name was Torben Krarup.

I do not know exactly when I met Torben personally for the first time. It might have been at the IUGG General Assembly 1967 in Lucerne. Anyway, there I became Chairman of the IAG Study Group on Mathematical Methods in Physical Geodesy, and Torben became a member of this group. I limited myself to suggesting some topics for future research, prominently among them least-squares prediction, Molodensky's boundary-value problem and the convergence of the spherical-harmonics series expansion of the external gravity potential at the Earth's surface.

Torben was by far the most active member of this study group. Already in 1968 he wrote a circular letter to the members, which I did not understand at all on first reading. Its title was "A framework for least-squares determination of the potential of the earth".

Least-Squares Collocation

In 1969 there appeared a small booklet as a publication of the Danish Geodetic Institute: "A contribution to the mathematical foundation of physical geodesy" by Torben Krarup. This I could understand. (It is reprinted in the 2006 Springer book "Mathematical Foundation of Geodesy" which Kai Borre compiled from Torben's papers with infinite care.) The above-mentioned "framework of least-squares determination" later became famous by the name of "least-squares collocation". It turned out to be not only an achievement of basic conceptual importance (this was the main concern of Torben Krarup) but also a wonderful numerical method for data combination. This was first shown by Christian Tscherning. I worked hard to make the method accessible to the general geodetic community. In fact, the battle about collocation raged among geodesists between 1970 and 1980. The main point of discussion was its statistical interpretation. Its origin was the least-squares prediction of stochastic processes developed by the famous American mathematician Norbert Wiener and the even more famous Russian mathematician Andrei Kolmogorov during World War 2, of course independently as a war secret.

The statistical treatment of the irregular gravity field in geodesy was started by R. A. Hirvonen and William Kaula in the years 1957–1962. When I came to Ohio State University in 1962, I began to apply the Kolmogorov-Wiener prediction to the interpolation and extrapolation (prediction) of gravity. In his doctor's dissertation (1964) Richard Rapp applied it numerically. I remember when he once came to me saying: "Helmut, your theory is wrong, I get imaginary error variances". I looked at his formulas, and said: "Dick, you have been using a non-positive covariance function". Everyone uses polynomials to approximate functions, but polynomials are not positive definite! This shows that seemingly subtle mathematical distinctions may be of great practical significance.

The ingenious idea of Torben Krarup was that least-squares prediction could be applied, not only to homogeneous data such as gravity anomalies, but also to heterogeneous data such as in combining gravity anomalies, deflections of the vertical, satellite data, etc. The matrix algorithm remains the same, only the quantities of the gravity field must be computed so as to be internally consistent ("covariance propagation").

This plays a role in criticizing the use of a basic statistical covariance function. But in fact, any initial analytical "kernel" function (provided it is positive-definite!) can be used, as Torben had remarked already in his fundamental 1969 booklet, but statistical kernel functions are frequently more convenient in practice, *representing general features of the gravity field*. Thus the covariance function of the gravity field is used as kernel function, and *the essential mathematical structure is ensured by the "covariance propagation"* (or "kernel propagation").

Error propagation plays a basic role in ordinary least-squares adjustment. It is a linear relation between error vectors. In the same way, covariance propagation is a linear propagation of "linear functionals" which are vectors in the infinite-dimensional Hilbert space of gravity. As has been already mentioned, the exact mathematical structure is assured provided the covariance propagation is done rigorously. Statistics does not spoil this mathematical structure, nor do noisy data.

A note on terminology. Why is this method called "collocation"? In numerical mathematics, *interpolation* is an approximation of an unknown function by fitting an approximate function (say, a polynomial) to *discrete measured values* of the unknown function. Fitting an approximate function (say, a polynomial) to *discrete measured functionals* (derivatives, integrals, ...) of the kernel function, has been called *"collocation"*. Where this name came in mathematics, I do not know. This is standard terminology in mathematics, and Torben Krarup, having been an excellent mathematician, knew of this terminology and used it in this abstract sense. A later, more concrete, geodetic attempt of interpretation was to derive collocation from the Latin verb "collocare", which means bringing things together (*con*) at one place (*locus*), collecting and combining things. In our case this would mean "combining and adjusting various geodetic measurements". This interpretation is more intuitive to a practical geodesist than the original mathematical meaning. So one may use both interpretations, or just use collocation for the method of Krarup, without bothering where the name came from.

Integrated geodesy. Least-squares collocation forms the nucleus of a theory of general data combination, later called "integrated geodesy" by Krarup. Today all this is usually simply called "collocation", which has become a household word in geodesy.

Convergence of Spherical-Harmonic Series

In his booklet, Torben also solved the problem of convergence or divergence of a spherical-harmonic development of the Earth's external gravitational potential at the Earth's surface. This was an old discussion among important people, which Krarup definitively solved by showing that it is a "non-problem". Even if this series were originally convergent, it could be made divergent by an arbitrary small change of the potential (the well-known "sand grain"). This I had known since 1962.

Which was much more important and difficult to prove, was that *the opposite is also true.* By an arbitrarily small change, the potential at the Earth surface can be made convergent, even if the original potential expression was divergent. In mathematical terms, the set of convergent potentials were *dense* in the set of all potentials, in much the same sense as the set of rational numbers are *dense* within the set of real numbers. Measurements are always finite, with a definite number of reliable digits, and can thus always be considered *rational* numbers. In the same way, measured spherical-harmonic series can always be considered convergent.

By realizing this, Torben Krarup made a giant contribution to geodetic thinking. In my opinion, this was his greatest contribution, because he proved the convergence problem to be a non-problem. Still, few people have understood this, and convergence discussions still flare up from time to time.

Krarup called this Runge's theorem, although Runge had proved it only for analytical functions of a complex variable, which are harmonic in the Gaussian plane. Krarup proved it for harmonic functions in three-dimensional space and still modestly called it "Runge's theorem". (This was good so because the spherical-harmonic case had already been considered in the book Frank – Mises, Die Differential- und Integralgleichungen der Mathematik und Physik, vol. 1 (1930). But without Torben we would never have known about it!)

The convergence problem is closely related to the problem of singularities in the analytical downward continuation of the external gravitational potential. In least-squares collocation we get finite linear combinations of the measuring data which are automatically *regular*.

Molodensky's Problem

Torben Krarup did not like to publish and still less to lecture. He preferred to work through informal discussion among friends, and through letters to a closed circle of study group members. He was a lecturer at two International Summer Schools in Mountains of Styria (1973 and 1975) and attended many of the Hotine-Marussi Symposia in Italy, because he found his proper atmosphere there and much liked Italy and his culture.

In his 1969 booklet he had an important section on an exact linearization of Molodensky's boundary problem, which is the problem of determining the Earth's surface from the gravity vector and the potential obtained by levelling. It is an extremely "hard" inverse problem in nonlinear analysis. The first step, linearization, was treated in an unusually careful mathematical manner, which was understandable also to pure mathematicians. He continued this in his extremely important *Four letters on Molodensky's problem* circulated in 1973 to the members of the above-mentioned Study Group on Mathematical Methods in Physical Geodesy. They are now in print available for the first time in Kai Borre's book, as well as his 1969 booklet, for which all of us should be grateful.

Arne Bjerhammar was able to interest the famous Swedish mathematician Lars Hörmander in Krarup's mathematical approach, and Hörmander gave an existence proof for Molodensky's boundary-value problem, though on very idealized mathematical assumptions. Anyway, it was a basic breakthrough.

When Fernando Sansò was in Graz for post-doctoral work in 1976, I gave a special small course on modern theoretical work. After my lecturing about Krarup's work on the linear Molodensky theory, he came to me afterwards, saying "One of these linear equations can be made to be exactly valid also for a certain non-linear version of the geodetic boundary-value problem"—and this was the birth of Sansò's well-known theory of the gravity space.

Conclusion

Least-squares collocation and Molodensky's problem were the two main areas of my cooperation with Torben Krarup and his Danish colleagues. Our relation can be compared with that of the Teacher and his Prophet. Torben was extremely creative but quite difficult to understand, and I considered my role mainly in putting all our ideas into a detailed form understandable to the international geodetic community, in my book "Advanced Physical Geodesy", Wichmann 1980.

I am not qualified to speak about Krarup's great ideas on geodetic networks developed together with Kai Borre and with my colleague Peter Meissl in Graz, who died much too early in his beloved Austrian Mountains.

Torben Krarup has been one of the most creative thinkers in mathematical geodesy in our age, and the Danish Geodetic Institute showed its wisdom to let him follow his thoughts and to provide an ideal scientific environment.

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