Autocorrelation Functions in GPS Data Processing: Modeling Aspects

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Consider a process that is actually random walk but is *incorrectly* modeled as a random constant $c$. We have then for the true model

$$x_k = x_{k-1} + \epsilon_k,$$

$$\epsilon_k = \text{unity Gaussian white noise},$$

$$\sigma^2\{x_0\} = 1$$

$$b_k = x_k + e_k,$$

$$k = 0, 1, 2, \ldots$$

$$\sigma^2\{e_k\} = 0.1$$

and for the incorrect model

$$x_k = c,$$

$$c \sim N(0, 1)$$

$$b_k = x_k + e_k,$$

$$k = 0, 1, 2, \ldots$$

$$\sigma^2\{e_k\} = 0.1$$

The wrong model has $F_k = 1$, $\Sigma_{\epsilon,k} = 0$, $A_k = 1$, $\Sigma_{e,k} = 0.1$, $\hat{x}_0 = 0$, and $P_0^{-1} = 1$. For the true model the parameters are the same except that $\Sigma_{\epsilon,k} = 1$, rather than zero.
Correctly and incorrectly filtered random walk process
Computing the Autocorrelation Function

Shift = 0:
\[
\begin{array}{cccccc}
    a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\
    a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\
\end{array}
\]

\[
\text{auto}(0) = \frac{1}{n-1} \sum_{i=0}^{n-1} a_i a_i / n.
\]

We multiply elements \(a_i a_i\) above each other, and add. Now shift the lower row:

Shift = 1:
\[
\begin{array}{cccccc}
    a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\
    a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\
\end{array}
\]

\[
\text{auto}(1) = \frac{1}{n-1} \sum_{i=1}^{n-1} a_i a_{i-1} / n.
\]

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Autocorrelation for random data. The peak at zero measures the variance.

- Linear trend subtracted
- Quadratic trend subtracted

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Ionospheric delays for one-ways, single and double differences

\[ I_k = \frac{(\Phi_{2,k} - \lambda_2 N_2) - (\Phi_{1,k} - \lambda_1 N_1)}{1 - (f_1/f_2)^2} \]
Autocorrelation for ionospheric delay to PRN 2. Upper left shows difference in time: \( I_k - I_{k-1} \), upper right the one-way, lower left the single difference, and lower left the double difference. Note the various orders of magnitude.
Autocorrelation of ionosphere delay for one-ways

<table>
<thead>
<tr>
<th>PRN</th>
<th>Elevation (in °)</th>
<th>$\sigma_I$ (in m)</th>
<th>Shift for first zero</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>master</td>
<td>rover</td>
</tr>
<tr>
<td>26</td>
<td>68.9</td>
<td>0.08</td>
<td>0.11</td>
</tr>
<tr>
<td>2</td>
<td>59.0</td>
<td>0.08</td>
<td>0.04</td>
</tr>
<tr>
<td>27</td>
<td>28.0</td>
<td>0.39</td>
<td>0.35</td>
</tr>
<tr>
<td>16</td>
<td>22.8</td>
<td>0.77</td>
<td>0.71</td>
</tr>
<tr>
<td>23</td>
<td>20.4</td>
<td>0.17</td>
<td>0.19</td>
</tr>
<tr>
<td>9</td>
<td>18.5</td>
<td>0.48</td>
<td>0.19</td>
</tr>
</tbody>
</table>
Reference

Kalman Filters, Correlation Functions, Power Spectrums, and Ambiguity Experiments

Kai Borre, Aalborg University
The following matrix equation

\[
\begin{bmatrix}
P_1 \\ \Phi_1 \\ P_2 \\ \Phi_2
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & -1 & \lambda_1 & 0 \\
1 & (f_1/f_2)^2 & 0 & 0 \\
1 & -(f_1/f_2)^2 & 0 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
\rho \\ I \\ N_1 \\ N_2
\end{bmatrix} -
\begin{bmatrix}
e_1 \\ \epsilon_1 \\ e_2 \\ \epsilon_2
\end{bmatrix}.
\]

(1)

can be used in various ways. One method is to solve for the 4 unknowns on an epoch-to-epoch basis. An alternative is to employ (1) as an observation equation in a filter. In any case we are faced with an interesting problem: \( \rho \) and \( I \) are to be estimated as reals while \( N_1 \) and \( N_2 \) by definition are integers. This situation is difficult. One procedure is to solve for all 4 unknowns as were they reals. This gives what is called a float solution. Given the float solution we can use a smart method that estimates \( N_1 \) and \( N_2 \) as integers. This refined solution is known as the fixed solution. A popular method is the LAMBDA method described by Teunissen [4]. However, our Matlab code uses a less complicated method described in [5].

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Computed positions for a static receiver. The present observations are taken with active SA. The variation over time is mainly due to SA.

- $X, Y, Z$ components of receiver position relative to mean position
- Raw receiver positions relative to mean position

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The first 300 epochs are deleted due to bad data. During epochs 300–1050 the rover remains at rest, from epoch 1050 it is moving a bit. During epochs 1291–1380 it is moved in a circle with radius ca. 16 meters. Then it rests at the circumference in epochs 1380–1587 and then makes a random movement outside the circle in epochs 1615–1787. The difference in computed positions for a static (master) receiver and a moving (rover) receiver. Now the detailed picture of the actual movement becomes clear. The differential positions are good to within a few dm.

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Double differenced code and phase observations on $L_1$ from static receivers. The ambiguities $N_1$ are estimated as a \textit{float solution} based on observations from epochs 300–400. Next the vector components $(x, y, z)$ are computed and the vector length.

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The figure shows the innovation for the four satellites as computed with respect to the reference satellite, covariance function for a selected innovation component, and the corresponding power spectrum illustrating the color of the innovation noise.

Estimated vector, L1 observations only, float ambiguities

\[ \sigma_{\text{vector}} = 12.6 \text{ mm} \]
Float and fixed values of $L_1$ ambiguities:

<table>
<thead>
<tr>
<th>Float Value</th>
<th>Fixed Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-221,971.00$</td>
<td>$-221,970$</td>
</tr>
<tr>
<td>$664,919.67$</td>
<td>$664,918$</td>
</tr>
<tr>
<td>$163,531.36$</td>
<td>$163,531$</td>
</tr>
<tr>
<td>$1,198,426.78$</td>
<td>$1,198,429$</td>
</tr>
</tbody>
</table>

The fixed values result from a procedure given by Yang et al. [5]. The above table lists the estimated ambiguities. We observe that the rounded values of the float ambiguities deviate at most 2 from the fixed values.

The standard deviation of the length of the vector between master and rover is $\sigma_{\text{vector}} = 16.7$ mm for the float solution and $\sigma_{\text{vector}} = 8.3$ mm for the fixed solution. We see that the vector components drift over time when incorrect ambiguities are used. Fixing the ambiguities to the correct values yields an estimate of the vector components at cm-level or even at mm-level.

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Double differenced code and phase observations on $L_1$ from static receivers. Note that the vector length is easier to estimate than its single components $x$, $y$, $z$. 

Estimated vector, L1 observations only, fixed ambiguities

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The ambiguities $N_1$ are estimated as a **fixed solution** based on observations from epochs 300–400. Note the dramatic increase in accuracy by fixing the ambiguities! Note the nice peak of the covariance function. The innovations are close to white noise.

![Graph showing the estimated vector, innovations, norm, correlation function, and power spectrum.](image)

$$\sigma_{\text{vector}} = 8.59 \text{ mm}$$

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The same as Figure 3, but at epoch 250 we change an ambiguity with the amount of 1 cycle.

Estimated vector, L1 observations only, fixed ambiguities, one changed by 1.
Estimated vector, L1 observations only, fixed ambiguities, one changed by 1 cycle

Innov. [mm]

Norm [m]

Corr. fnc. [mm$^2$]

Pow. spec. [dB]

$\sigma_{\text{vector}} = 66.2$ mm

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The length of the baseline vector $\|v\| = \sqrt{x^2 + y^2 + z^2} = d$ often is easier to estimate than the components of $v = (x, y, z)$. We illustrate this by applying the law of variance propagation:

$$
\sigma_d^2 = \left[ \frac{\partial d}{\partial x} \frac{\partial d}{\partial y} \frac{\partial d}{\partial z} \right] \Sigma_v \left[ \begin{array}{c} \frac{\partial d}{\partial x} \\ \frac{\partial d}{\partial y} \\ \frac{\partial d}{\partial z} \end{array} \right] = \left( \frac{1}{d} v^T \right) \Sigma_v \left( \frac{1}{d} v \right).
$$

For the actual data and with fixed ambiguities we have $\sigma_d = 3.5$ mm, $\sigma_x = 4.8$ mm, $\sigma_y = 1.7$ mm, and $\sigma_z = 3.2$ mm. This partially supports our statement.
All $M$-files producing the figures in this paper can be found at kom.auc.dk/~borre/life-199

The original Ashtech data are stored in files *.105. The $M$-files $\texttt{bdata}$, $\texttt{edata}$, and $\texttt{sdata}$ read the binary observation files and rearrange the data into a useful MATLAB format in the *.dat files.

The call $\texttt{rawrecep}$ produces our first figure. This file calls two other $M$-files $\texttt{togeod}$ and $\texttt{topocent}$. $\texttt{togeod}$ is a function that computes geodetic coordinates ($\phi$, $\lambda$, $h$) given Cartesian coordinates ($X, Y, Z$) and the semi-major axis $a$ and inverse flattening $f$ for a reference ellipsoid. $\texttt{topocent}$ transforms a 3-D vector $d\,x$ into topocentric coordinates ($\text{Dist}$, $\text{Az}$, $\text{El}$) with origin at $X$.

The main $M$-file $\texttt{makebase}$ reads the *.dat files, estimates ambiguities for DD on L1 according to Yang et al. (1994) and uses $\texttt{get\_eph}$ and $\texttt{find\_eph}$ to compute satellite positions. The file $\texttt{get\_rho}$ computes the distance in ECEF between satellite and receiver at raw transmit time given an ephemeris. Now we can write the observation equations for the unknown vector components ($x$, $y$, $z$). We solve the normals on an epoch-to-epoch basis.

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Conclusions

We have given a standard presentation of GPS observations and their characteristics. With a real data set we focus on a few specific topics: float and fixed ambiguity solutions and the effect on the estimated vector. In our special case we note that the standard deviation of the vector length is halved when changing from float to fixed solution. We also note that it is easier to estimate the vector length rather than the vector components themselves. Under controlled circumstances we change an ambiguity and study the effect on the solution and observe how the innovation vector slowly strives for the correct known solution.
References


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Block Elimination and Weight Matrices

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Block Elimination

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}.
\]

\[
\begin{bmatrix}
I & 0 \\
-CA^{-1} & I
\end{bmatrix}
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
I & 0 \\
-CA^{-1} & I
\end{bmatrix}
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix}
\]

or explicitly

\[
\begin{bmatrix}
A & B \\
0 & D - CA^{-1}B
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
= 
\begin{bmatrix}
b_1 \\
b_2 - CA^{-1}b_1
\end{bmatrix}.
\]

Final result

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}^{-1}
= 
\begin{bmatrix}
A^{-1}(I + B(D - CA^{-1}B)^{-1}CA^{-1}) & -A^{-1}B(D - CA^{-1}B)^{-1} \\
-(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1}
\end{bmatrix}.
\]
Schreiber’s Device

In 1877 Schreiber described a procedure for eliminating the unknowns \( x_1 \) (we present a formulation somewhat more general than the original one):

Let be given a set of observation equations \( A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b - \epsilon \). The unknown \( x_1 \) is eliminated if we delete the columns in \( A \) corresponding to the unknown \( x_1 \), and add the fictitious observational row

\[
\begin{bmatrix}
a^T_1 \Sigma^{-1} a_2 & a^T_1 \Sigma^{-1} a_3 & \ldots & a^T_1 \Sigma^{-1} a_n
\end{bmatrix}
\]

having the negative weight \(-1/a^T_1 \Sigma^{-1} a_1\).
We demonstrate the procedure on a numerical example:

\[
\begin{bmatrix}
1 & 0 \\
1 & 1 \\
1 & 3 \\
1 & 4 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\end{bmatrix}
=
\begin{bmatrix}
0 \\
8 \\
8 \\
20 \\
\end{bmatrix}.
\]

The augmented left side of the normals $A^T \Sigma^{-1} A$ is

\[
\begin{bmatrix}
0 & 1 & 3 & 4 & 8 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
3 \\
4 \\
\end{bmatrix}
=
\begin{bmatrix}
10 \\
\end{bmatrix}.
\]

The augmented right side of the normals $A^T \Sigma^{-1} b$ is

\[
\begin{bmatrix}
0 & 1 & 3 & 4 & 8 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
0 \\
8 \\
8 \\
20 \\
\end{bmatrix}
=
\begin{bmatrix}
40 \\
\end{bmatrix}.
\]

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Partition of Observation Equations

Let be given a set of observation equations $Ax = b - \epsilon$ partitioned into

$$
\begin{bmatrix}
A_1 \\
A_2 
\end{bmatrix} x =
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} -
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2
\end{bmatrix}
$$

with weights

$$
\begin{bmatrix}
\Sigma_1^{-1} \\
\Sigma_2^{-1}
\end{bmatrix}.
$$

(2)

The matrix $A_1$ describes usual geodetic observations while $A_2$ exclusively deals with observations of coordinates. Considering only the first block row of observations we get the usual solution $\hat{x}_1 = \hat{x}_f$:

$$
\hat{x}_f = (A_1^T \Sigma_1^{-1} A_1)^{-1} A_1^T \Sigma_1^{-1} b_1 = \Sigma_f A_1^T \Sigma_1^{-1} b_1.
$$

The subscript $f$ stands for free: The $\hat{x}_f$ solution relates to a network not constrained to the observed coordinates $b_2$. Of course we suppose $\Sigma_f$ exists!

Next we look for the solution $x_c$ when we include the second block row, namely the coordinate observations $A_2 x = b_2 - \epsilon_2$. By adding the normals from both rows we get
\begin{equation}
(A_1^T \Sigma_1^{-1} A_1 + A_2^T \Sigma_2^{-1} A_2) \hat{x} = A_1^T \Sigma_1^{-1} b_1 + A_2^T \Sigma_2^{-1} b_2. \tag{3}
\end{equation}

We recall that \( \Sigma_f^{-1} = A_1^T \Sigma_1^{-1} A_1 \) and \( A_1^T \Sigma_1^{-1} b_1 = \Sigma_f^{-1} \hat{x}_f \):

\[
\hat{x}_c = (\Sigma_f^{-1} + A_2^T \Sigma_2^{-1} A_2)^{-1} (\Sigma_f^{-1} \hat{x}_f + A_2^T \Sigma_2^{-1} b_2).
\]

The common step in all derivations is our favorite matrix identity

\[
\Sigma = (T + A^T \Sigma_e^{-1} A)^{-1} = T^{-1} - T^{-1} A^T (\Sigma_e + A T^{-1} A^T)^{-1} A T^{-1}. \tag{4}
\]

With (4) we get

\[
\hat{x}_c = (\Sigma_f - \Sigma_f A_2^T (\Sigma_2 + A_2 \Sigma_f A_2^T)^{-1} A_2 \Sigma_f) (\Sigma_f^{-1} \hat{x}_f + A_2^T \Sigma_2^{-1} b_2).
\]

Multiplying out yields

\[
\hat{x}_c = \hat{x}_f + \Sigma_f A_2^T \Sigma_2^{-1} b_2 - \Sigma_f A_2^T (\Sigma_2 + A_2 \Sigma_f A_2^T)^{-1} A_2 \hat{x}_f \\
- \Sigma_f A_2^T (\Sigma_2 + A_2 \Sigma_f A_2^T)^{-1} A_2 \Sigma_f A_2^T \Sigma_2^{-1} b_2.
\]
Rearranging we get

\[ \hat{x}_c = \hat{x}_f - \Sigma_f A_2^T (\Sigma_2 + A_2 \Sigma_f A_2^T)^{-1} A_2 \hat{x}_f \]
\[ + (\Sigma_f - \Sigma_f A_2^T (\Sigma_2 + A_2 \Sigma_f A_2^T)^{-1} A_2 \Sigma_f) A_2^T \Sigma_2^{-1} b_2. \]

Using (4) once again—in the opposite direction—we get

\[ \hat{x}_c = \hat{x}_f - \Sigma_f A_2^T (\Sigma_2 + A_2 \Sigma_f A_2^T)^{-1} A_2 \hat{x}_f + (\Sigma_f^{-1} + A_2^T \Sigma_2^{-1} A_2)^{-1} A_2^T \Sigma_2^{-1} b_2. \]

Finally the matrix identity (26) takes us the last step:

\[ \hat{x}_c = \hat{x}_f - \Sigma_f A_2^T (\Sigma_2 + A_2 \Sigma_f A_2^T)^{-1} A_2 \hat{x}_f + \Sigma_f A_2^T (\Sigma_2 + A_2 \Sigma_f A_2^T)^{-1} b_2 \]

or rearranged

\[ \hat{x}_c = \hat{x}_f + \Sigma_f A_2^T (\Sigma_2 + A_2 \Sigma_f A_2^T)^{-1} (b_2 - A_2 \hat{x}_f). \]  

(5)

From (3) we read the covariance matrix of the \textit{constrained} estimate \( \hat{x}_c \)

\[ \Sigma_c = (A_1^T \Sigma_1^{-1} A_1 + A_2^T \Sigma_2^{-1} A_2)^{-1}. \]  

(6)

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We recall that $\Sigma_f = (A_1^T \Sigma_1^{-1} A_1)^{-1}$, so (6) can be written

$$\Sigma_c = (\Sigma_f^{-1} + A_2^T \Sigma_2^{-1} A_2)^{-1}. \quad (7)$$

Using (4) we get

$$\Sigma_c = \Sigma_f - \Sigma_f A_2^T (\Sigma_2 + A_2 \Sigma_2 A_2^T)^{-1} A_2 \Sigma_f. \quad (8)$$

A nonnegative definite matrix is subtracted from $\Sigma_f$, so

$$\Sigma_c \leq \Sigma_f. \quad (9)$$

This says that covariances in the constrained network are smaller than or equal to those in the free network.

Putting $\Sigma_2 = 0$ in (2)—the observation error equals zero—is equivalent to introduce the constraint $A_2 x = b_2$:

$$A_1 x = b_1 - \epsilon_1 \quad (10)$$

$$A_2 x = b_2. \quad (11)$$
This has a large impact on formulation of filters. Krarup calls this a *hard postulation* of the least-squares problem, opposed to a *soft postulation* of the constraints with $\Sigma_2 > 0$.

*So setting the observation error equal to zero in a Kalman filter is the same as using a constraint in a standard least-squares formulation.*
Dependency on the Weights

\[ b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}, \quad A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad \Sigma^{-1} = \begin{bmatrix} \Sigma_1^{-1} & 0 \\ 0 & \Sigma_2^{-1} \end{bmatrix}. \]

The normal equation for the original, total problem is

\[
(A_1^T \Sigma_1^{-1} A_1 + A_2^T \Sigma_2^{-1} A_2) \hat{x} = A_1^T \Sigma_1^{-1} b_1 + A_2^T \Sigma_2^{-1} b_2. \tag{12}
\]

The perturbed problem is

\[
(A_1^T \Sigma_1^{-1} A_1 + A_2^T (\Sigma_2^{-1} + (\Delta \Sigma_2^{-1})) A_2) (\hat{x} + \Delta x) = \\
(A_1^T \Sigma_1^{-1} b_1 + A_2^T (\Sigma_2^{-1} + (\Delta \Sigma_2^{-1})) b_2). \tag{13}
\]

Now subtract (12) from (13) to find an equation for \( \Delta x \):

\[
(A_1^T \Sigma_1^{-1} A_1 + A_2^T (\Sigma_2^{-1} + (\Delta \Sigma_2^{-1})) A_2) \Delta x + A_2^T (\Delta \Sigma_2^{-1}) A_2 \hat{x} = A_2^T (\Delta \Sigma_2^{-1}) b_2.
\]
We set \( N = A_1^T \Sigma_1^{-1} A_1 + A_2^T \Sigma_2^{-1} A_2 \) and \( \hat{e}_2 = b_2 - A_2 \hat{x} \). Then the change in \( \hat{x} \) is

\[
\Delta x = (N + A_2^T (\Delta \Sigma_2^{-1}) A_2)^{-1} A_2^T (\Delta \Sigma_2^{-1}) \hat{e}_2. \tag{14}
\]

Now (4) yields the change in the inverse:

\[
(N + A_2^T (\Delta \Sigma_2^{-1}) A_2)^{-1} = N^{-1} - N^{-1} A_2^T ((\Delta \Sigma_2^{-1})^{-1} + A_2 N^{-1} A_2^T)^{-1} A_2 N^{-1}. \]

For \( m \) observations, this matrix multiplies \( A_2^T (\Delta \Sigma_2^{-1}) \hat{e}_2 \) to give \( \Delta x \). If we specialize to one single observation, \( a_2^T \) is an \( m \) by 1 vector and \( \Delta \Sigma_2^{-1} \) is a 1 by 1 scalar. We name the product \( a_2^T N^{-1} a_2 = s \) and get

\[
\Delta x = N^{-1} a_2^T \left( 1 - \frac{\Delta \Sigma_2^{-1}}{1 + s (\Delta \Sigma_2^{-1})} \right) (\Delta \Sigma_2^{-1}) \hat{e}_2
\]

or

\[
\Delta x = \frac{\hat{e}_2 (\Delta \Sigma_2^{-1})}{1 + s (\Delta \Sigma_2^{-1})} N^{-1} a_2^T. \tag{15}
\]
A First Order Derivation

If we consider a small change $\Delta \Sigma^{-1}$ in the weighting matrix, we can compute the corresponding small movement $\Delta \hat{x}$ in the estimate. The problem is linearized and higher order terms are ignored:

$$A^T(\Sigma^{-1} + (\Delta \Sigma^{-1}))A(\hat{x} + \Delta \hat{x}) = A^T(\Sigma^{-1} + (\Delta \Sigma^{-1}))b.$$  

Subtracting $A^T\Sigma^{-1}A\hat{x} = A^T\Sigma^{-1}b$ leaves an equation for the first-order correction $\Delta \hat{x}$:

$$A^T\Sigma^{-1}A(\Delta \hat{x}) = A^T(\Delta \Sigma^{-1})(b - A\hat{x}).$$

This correction to $\hat{x}$ is small when $\Delta \Sigma^{-1}$ is small, and when the residual $b - A\hat{x}$ is small.

If we work with the covariance matrix $\Sigma$ instead of $\Sigma^{-1}$, linearization gives $\Delta \Sigma^{-1} = -\Sigma^{-1}(\Delta \Sigma)\Sigma^{-1}$. This is the matrix analog of $-\Delta x/x^2$, the first-order change in $1/x$.

Linearization also gives the change in $Q = (A^T\Sigma^{-1}A)^{-1}$, which contains the variances and covariances of the estimate $\hat{x}$:

$$\Delta Q = QA^T(\Delta \Sigma^{-1})AQ.$$  

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**Useful Matrix Identities**

Whenever the following inverse matrices exist we have

\[(AB)^{-1} = B^{-1}A^{-1}\] \hspace{1cm} (16)

\[(I + AB)^{-1}A = A(I + BA)^{-1}\] \hspace{1cm} (17)

\[A \text{ and } B \text{ may not be square}\] \hspace{1cm} (18)

for vectors we especially have

\[(I + ab^T)^{-1}a = a(I + b^Ta)^{-1}\] \hspace{1cm} (19)

\[(A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B = B(A + B)^{-1}A\] \hspace{1cm} (20)

\[(A + BDC)^{-1} = A^{-1} - A^{-1}B(D^{-1} + CA^{-1}B)^{-1}CA^{-1}\] \hspace{1cm} (21)

\[DC(A + BDC)^{-1} = DCA^{-1}(I + BDCCA^{-1})\] \hspace{1cm} (22)

\[= DC(I + A^{-1}BDC)^{-1}A^{-1}\] \hspace{1cm} (23)

\[= D(I + CA^{-1}BD)^{-1}CA^{-1}\] \hspace{1cm} (24)

\[= (I + DCA^{-1}B)^{-1}DCA^{-1}\] \hspace{1cm} (25)

\[= (D^{-1} + CA^{-1}B)^{-1}CA^{-1}.\] \hspace{1cm} (26)

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MATLAB Code Demonstrating the Theory

\[
A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 3 & 7 \\ 3 & 4 & 8 \\ 4 & 5 & 10 \end{bmatrix}; \\
b = [0;7;8;20]; \\
x = A \backslash b \\
A1 = A(:,1:2); \\
A2 = A(:,3); \\
\%Projector \\
P = \text{eye}(4) - A1 \cdot \text{inv}(A1' \cdot A1) \cdot A1'; \\
PA = P \cdot A; \\
Pb = P \cdot b; \\
x3 = Pb(4)/PA(4,3) \\
\%Block elimination \\
N = A' \cdot A; \\
b0 = A' \cdot b; \\
b1 = b0(1:2); \\
b2 = b0(3); \\
A0 = N(1:2,1:2); \\
B = N(1:2,3); \\
D = N(3,3); \\
x3 = \text{inv}(D - B' \cdot \text{inv}(A0) \cdot B) \cdot (b2 - B' \cdot \text{inv}(A0) \cdot b1) \\
\%General formula \\
M = A2' \cdot A1 \cdot \text{inv}(A1' \cdot A1) \cdot A1'; \\
R = A2' - M; \\
x3 = \text{inv}(R \cdot A2) \cdot R \cdot b
\]
Reference