Positioning Based on GPS Pseudoranges

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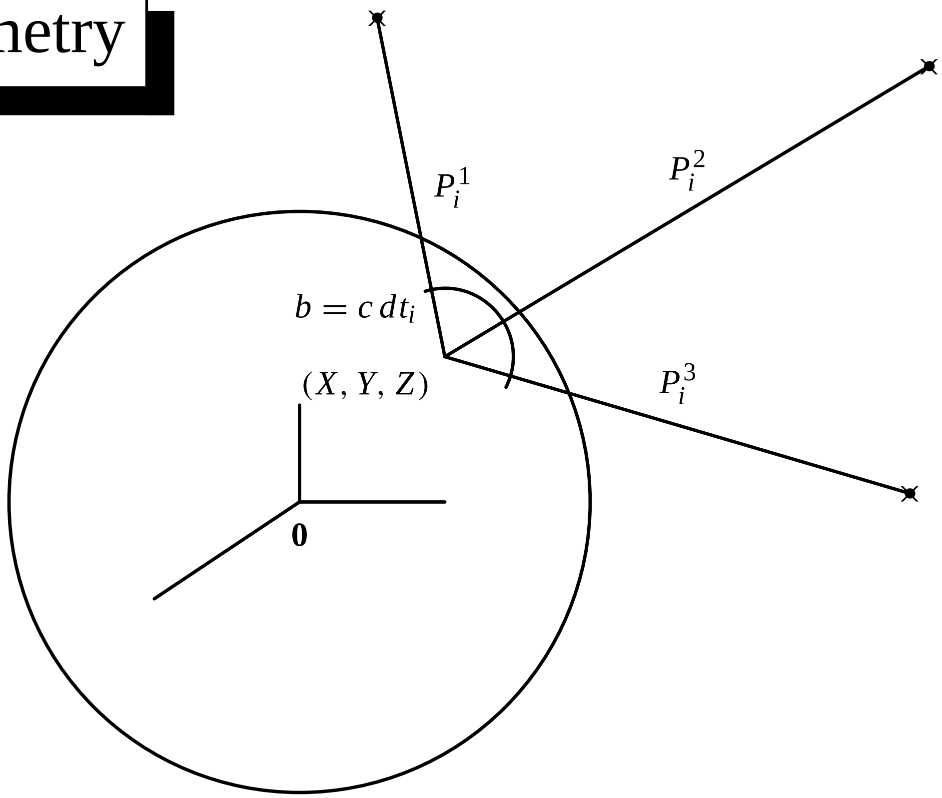
Observation Geometry



The ranges to GPS satellites measured by a receiver have a common bias and, therefore, are called *pseudo*ranges:

$$P_i^k = \rho_i^k + c\,dt_i + \text{noise}.$$

Observation Equation

Pseudorange on L_1 frequency; signal travel time $\tau_i^k = t_i - t^k$:

$$P_i^k = \rho_i^k + c\,dt_i(t) - c\,dt^k(t - \tau_i^k) + I_i^k + T_i^k - e_i^k. \qquad (1)$$

The geometric distance between satellite k and receiver i is $\rho_i^k = \sqrt{(X^k - X_i)^2 + (Y^k - Y_i)^2 + (Z^k - z_i)^2}$. Neglecting satellite clock offset, ionospheric and tropospheric delays we get for $k = 1, 2, 3, 4$

$$\begin{bmatrix} P_i^1 \\ P_i^2 \\ P_i^3 \\ P_i^4 \end{bmatrix} = \begin{bmatrix} \rho_i^1 + c\,dt_i \\ \rho_i^2 + c\,dt_i \\ \rho_i^3 + c\,dt_i \\ \rho_i^4 + c\,dt_i \end{bmatrix}.$$

The distance between satellite k and receiver i—corrected for Earth rotation, rotation rate of the Earth is ω_e—is defined by

$$\rho_i^k = \| R_3(\omega_e \tau_i^k) \boldsymbol{r}^k (t - \tau_i^k)_{\text{geo}} - \boldsymbol{r}_i(t) \|.$$

The matrix R_3 accounts for the rotation by the angle $\omega_e \tau_i^k$ while the signal is traveling:

$$R_3(\omega_e \tau_i^k) = \begin{bmatrix} \cos(\omega_e \tau_i^k) & \sin(\omega_e \tau_i^k) & 0 \\ -\sin(\omega_e \tau_i^k) & \cos(\omega_e \tau_i^k) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The rotation is necessary when using vectors referenced to an Earth centered and Earth fixed system (ECEF).

We linearize (1) and let ^{0} denote a preliminary value:

$$-\frac{X^k - X_i^0}{\left(\rho_i^k\right)^0} x_i - \frac{Y^k - Y_i^0}{\left(\rho_i^k\right)^0} y_i - \frac{Z^k - Z_i^0}{\left(\rho_i^k\right)^0} z_i + 1(c\,dt_i)$$

$$= P^k_{i\,\mathrm{obs}} - \left(P_i^k\right)^0 - e_i^k = b_i - e_i^k. \qquad (2)$$

The unknowns are arranged as $\boldsymbol{x} = (x_i, y_i, z_i, c\,dt_i)$ and we get

$$A\boldsymbol{x} = \begin{bmatrix} -\frac{X^1 - X_i^0}{\rho_i^1} & -\frac{Y^1 - Y_i^0}{\rho_i^1} & -\frac{Z^1 - Z_i^0}{\rho_i^1} & 1 \\ -\frac{X^2 - X_i^0}{\rho_i^2} & -\frac{Y^2 - Y_i^0}{\rho_i^2} & -\frac{Z^2 - Z_i^0}{\rho_i^2} & 1 \\ & \vdots & & \\ -\frac{X^m - X_i^0}{\rho_i^m} & -\frac{Y^m - Y_i^0}{\rho_i^m} & -\frac{Z^m - Z_i^0}{\rho_i^m} & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \\ c\,dt_i \end{bmatrix} = \boldsymbol{b} - \boldsymbol{e}. \qquad (3)$$

Introducing the unit direction vector from receiver i to satellite k

$$(\boldsymbol{u}_i^k)^0 = \left(\frac{X_{\text{ECEF}}^k - X_i^0}{\rho_i^k}, \frac{Y_{\text{ECEF}}^k - Y_i^0}{\rho_i^k}, \frac{Z_{\text{ECEF}}^k - Z_i^0}{\rho_i^k} \right)$$

we can rewrite (3) as

$$A\boldsymbol{x} = \begin{bmatrix} -(\boldsymbol{u}_i^1)^0 & 1 \\ -(\boldsymbol{u}_i^2)^0 & 1 \\ \vdots & \\ -(\boldsymbol{u}_i^m)^0 & 1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{b} - \boldsymbol{e}.$$

The least-squares solution becomes

$$\begin{bmatrix} \hat{x}_i \\ \hat{y}_i \\ \hat{z}_i \\ \widehat{c\,dt_i} \end{bmatrix} = (A^{\mathrm{T}} \Sigma^{-1} A)^{-1} A^{\mathrm{T}} \Sigma^{-1} \boldsymbol{b}. \tag{4}$$

Often one uses $\Sigma = \sigma^2 I$.

After a few iterations the final receiver coordinates become $\widehat{X}_i = X_i^0 + \hat{x}_i$, $\widehat{Y}_i = Y_i^0 + \hat{y}_i$, and $\widehat{Z}_i = Z_i^0 + \hat{z}_i$.

A Posteriori Covariance Matrix and DOP

The a posteriori covariance matrix connected with the solution (4) is $\Sigma_{\text{ECEF}} = (A^{\text{T}}\Sigma^{-1}A)^{-1}$:

$$\Sigma_{\text{ECEF}} = \left[\begin{array}{ccc|c} \sigma_X^2 & \sigma_{XY} & \sigma_{XZ} & \sigma_{X,c\,dt} \\ \sigma_{YX} & \sigma_Y^2 & \sigma_{YZ} & \sigma_{Y,c\,dt} \\ \sigma_{ZX} & \sigma_{ZY} & \sigma_Z^2 & \sigma_{Z,c\,dt} \\ \hline \sigma_{c\,dt,X} & \sigma_{c\,dt,Y} & \sigma_{c\,dt,Z} & \sigma_{c\,dt}^2 \end{array}\right]. \tag{5}$$

The matrix F^{T} connects Cartesian coordinate differences in the local system (at latitude ϕ and longitude λ) and the ECEF system. The sequence (E, N, U)

assures that both the local and the ECEF systems shall be right handed:

$$F^{\mathrm{T}} = R_3(\pi)R_2\big(\phi - \tfrac{\pi}{2}\big)R_3(\lambda - \pi)$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\phi & 0 & \cos\phi \\ 0 & 1 & 0 \\ -\cos\phi & 0 & \sin\phi \end{bmatrix} \begin{bmatrix} -\cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & -\cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{bmatrix}. \qquad (6)$$

The covariance propagation law transforms Σ_{ECEF} into the covariance matrix expressed in a local system with coordinates (E, N, U). The interesting 3 by 3 submatrix S of Σ_{ECEF} is shown in (5). After the transformation F, the submatrix becomes

$$\Sigma_{\text{ENU}} = \begin{bmatrix} \sigma_E^2 & \sigma_{EN} & \sigma_{EU} \\ \sigma_{NE} & \sigma_N^2 & \sigma_{NU} \\ \sigma_{UE} & \sigma_{UN} & \sigma_U^2 \end{bmatrix} = F^{\mathrm{T}} S F. \tag{7}$$

In practice we meet several forms of the ***dilution of precision*** (abbreviated DOP):

Geometric: $$\text{GDOP} = \sqrt{\frac{\sigma_E^2 + \sigma_N^2 + \sigma_U^2 + \sigma_{c\,dt}^2}{\sigma_0^2}} = \sqrt{\frac{\operatorname{tr}(\Sigma_{\text{ECEF}})}{\sigma_0^2}}$$

Horizontal: $\mathrm{HDOP} = \sqrt{\frac{\sigma_E^2 + \sigma_N^2}{\sigma_0^2}}$

Position: $\mathrm{PDOP} = \sqrt{\frac{\sigma_E^2 + \sigma_N^2 + \sigma_U^2}{\sigma_0^2}} = \sqrt{\frac{\sigma_X^2 + \sigma_Y^2 + \sigma_Z^2}{\sigma_0^2}}$

Time: $\mathrm{TDOP} = \sigma_{c\,dt}/\sigma_0$

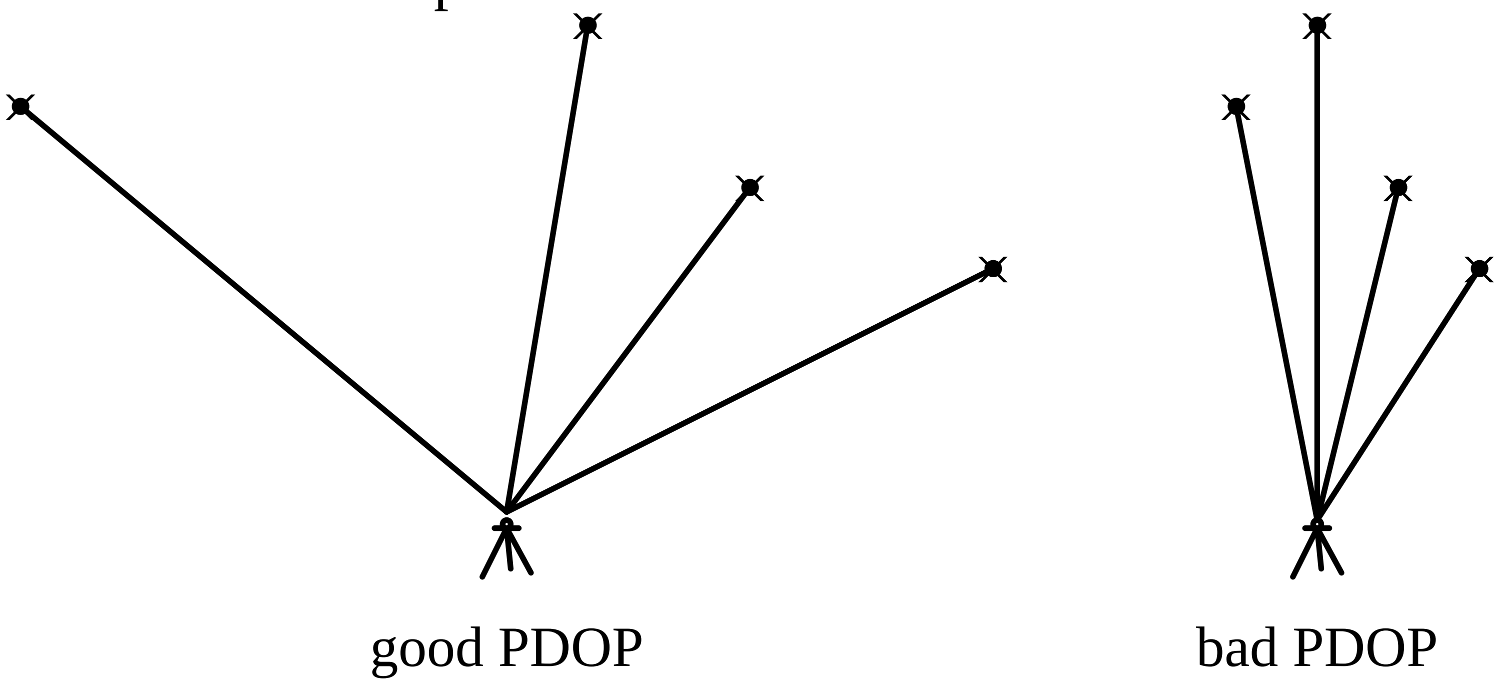
Vertical: $\mathrm{VDOP} = \sigma_U/\sigma_0.$

Note that all DOP values are dimensionless. They multiply the range errors to give the position errors (approximately). Furthermore we have

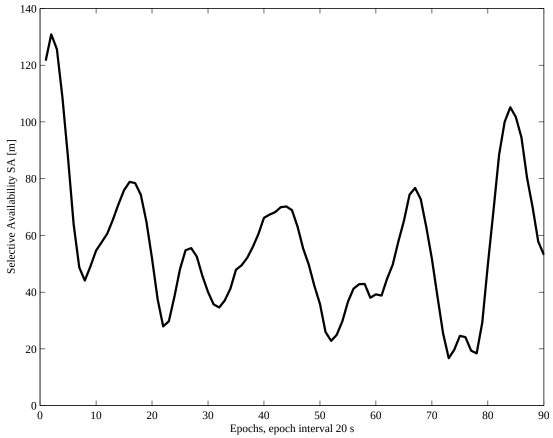
$$\mathrm{GDOP}^2 = \mathrm{PDOP}^2 + \mathrm{TDOP}^2 = \mathrm{HDOP}^2 + \mathrm{VDOP}^2 + \mathrm{TDOP}^2.$$

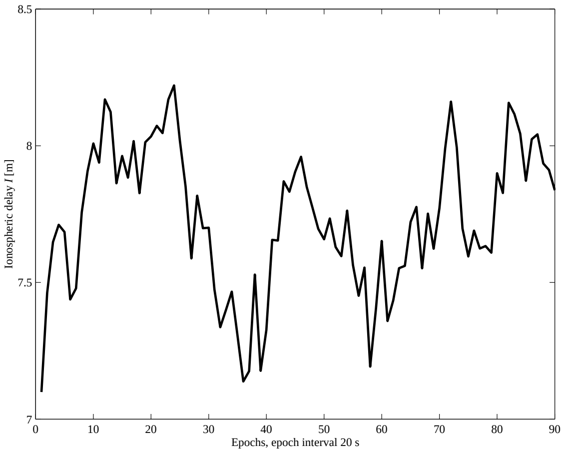
Geometric Interpretation of DOP

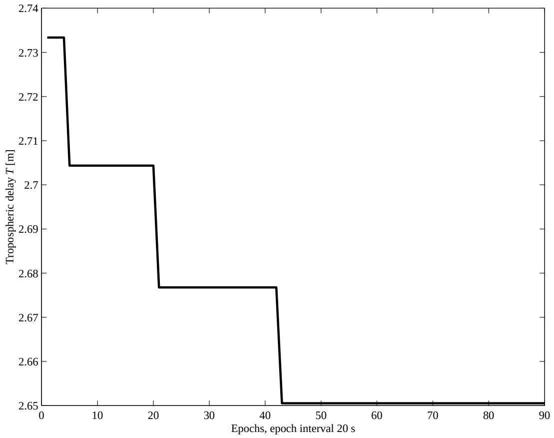
PDOP can be interpreted as the reciprocal value of the volume V of a tetrahedron that is formed by the satellites and the receiver: $\text{PDOP} = \frac{1}{V}$. The best geometric situation exists when PDOP is minimized. With a full satellite configuration, the concept of DOP has less impact.

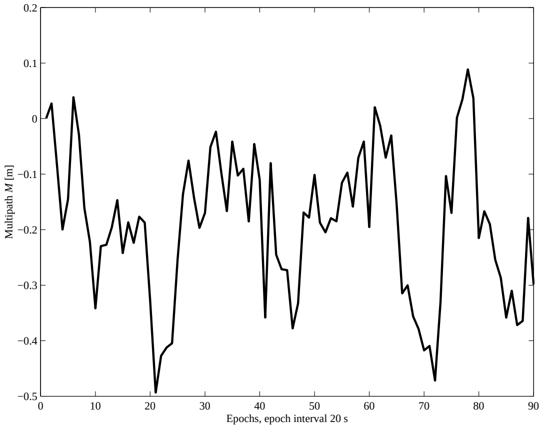


SA, Ionospheric and Tropospheric Delays, and Multipath

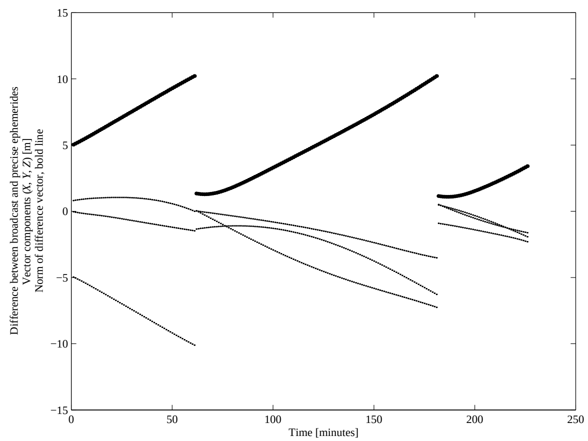


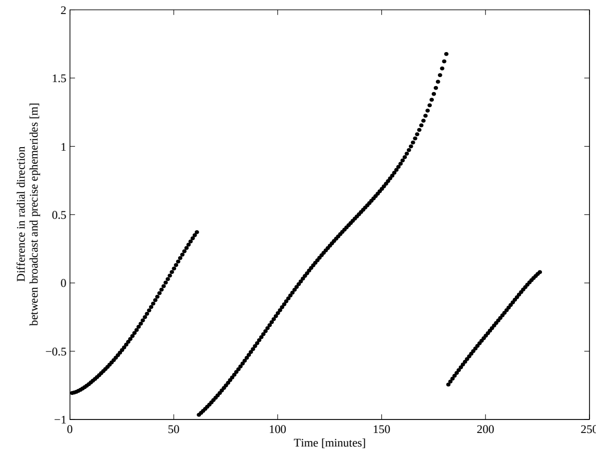






Orbit Errors





The left plot shows the difference between precise and broadcast ephemerides. The precise ephemerides are accurate to within 5 cm. The right plot shows the length of the difference vector between precise and broadcast ephemerides as projected onto the line of sight.

Modernization of GPS

Standard Positioning Service (SPS), σ in meters

Error Source	With SA	Without SA	C/A code on L2 and/or L5	With AII[a]
SA	24.0	0.0	0.0	0.0
Ionosphere	7.0	7.0	0.01	0.01
Troposphere	2.0	0.2	0.2	0.2
Orbit and Clock	2.3	2.3	2.3	1.25
Receiver Noise	0.6	0.6	0.6	0.6
Multipath	1.5	1.5	1.5	1.5
User Equivalent Range Error	25.0	7.5	2.8	2.0
HDOP	1.5	1.5	1.5	1.5
Stand-Alone, Horizontal Acc., 95%	**75.0**	**22.5**	**8.5**	**6.0**
Implementation Date		May 2, 2000	2003–2006	2005–2010

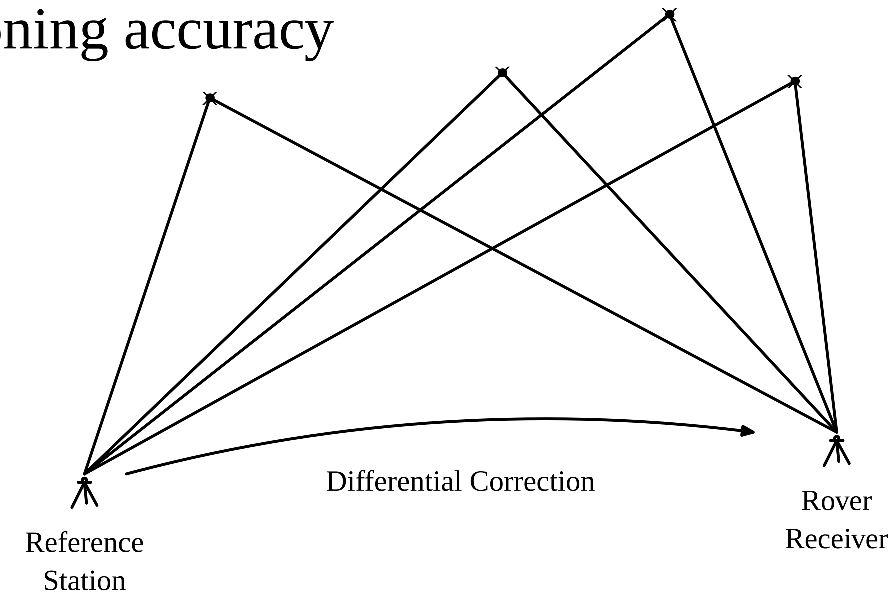
Operational Control Segment improvements from 2000

Source: GPS World, September 2000, pp. 36–44

[a] Accuracy Improvement Initiative

Geometry for Differential GPS

- occupy a known station
- compute range corrections
- transmit range corrections to rover
- corrections applied at rover
- improved positioning accuracy



Differential GPS

Assume the position of receiver i is ***known***. P_i^k is related to $c\,dt_i$ and can be applied as a range correction to P_j^k of the rover. Combining the models at i and j leads to the principle of ***differential GPS***:

$$\begin{bmatrix} \Big(P^1_{j\,\mathrm{obs}} - \big(P^1_j\big)^0\Big) - \Big(P^1_{i\,\mathrm{obs}} - \big(P^1_i\big)^0\Big) \\ \Big(P^2_{j\,\mathrm{obs}} - \big(P^2_j\big)^0\Big) - \Big(P^2_{i\,\mathrm{obs}} - \big(P^2_i\big)^0\Big) \\ \vdots \\ \Big(P^m_{j\,\mathrm{obs}} - \big(P^m_j\big)^0\Big) - \Big(P^m_{i\,\mathrm{obs}} - \big(P^m_i\big)^0\Big) \end{bmatrix} = \begin{bmatrix} -\big(\boldsymbol{u}^1_j\big)^0 & 1 \\ -\big(\boldsymbol{u}^2_j\big)^0 & 1 \\ \vdots & \\ -\big(\boldsymbol{u}^m_j\big)^0 & 1 \end{bmatrix} \begin{bmatrix} x_j \\ y_j \\ z_j \\ c\,dt_{ij} \end{bmatrix}.$$

Here $c\,dt_{ij} = c\,dt_j - c\,dt_i$ is the difference of receiver clock offsets.

Wide Area DGPS (WADGSP)

WADGPS provides a powerful means for bridging the gap between single site and high-accuracy positioning in the vicinity of a correction station:

1. Monitor stations at known locations collect GPS pseudoranges from all satellites in view

2. Pseudoranges and dual-frequency ionospehric delay measurements (if available) are sent to the master station

3. Master station computes an error correction vector

4. Error correction vector is transmitted to users

5. Users apply error corrections to their measured pseudoranges and collected ephemeris data to improve navigation accuracy.

Internet-based Global DGPS

NASA's Global GPS Network (GGN) is operated and maintained by JPL. GGN consists of more then 60 sites in batch mode over the Internet **igscb.jpl.nasa.gov**. Software used:

- Real-Time Net Transfer (RTNT)
- Real-Time GIPSY (RTG).

The open Internet is a reliable choice to return GPS data for a state-space global differential system. User positions accurate to sub-meter level.
gipsy.jpl.nasa.gov/igdg/system/index.html

Virtual Reference Station

Data from several reference stations are collected and processed in real-time using standard software.

Via the Internet the user asks for data from a non-exisiting reference station at a user specified location. The accuracy is comparable to the one obtained from a rigorous network solution.

Source: H. van der Marel (1998) *Virtual GPS Reference Stations in the Netherlands*. ION GPS-98, pp. 49–58

GPS for Precise Time

GPS is the primary system for distribution of Precise Time and Time Interval (PTTI). The time scales are

- time kept by a satellite clock t^k
- GPS time t_{GPS} defined by the Control Segment on the basis of a set of atomic standards aboard the satellites and in monitor stations
- UTC (USNO) t_{UTC} the US national standard defined by the US Naval Observatory
- time kept by a user's receiver clock t_i

$\delta t_{\mathrm{UTC}} = t_{\mathrm{GPS}} - t_{\mathrm{UTC}}$ is currently estimated to be about 10 ns.

Synchronization of clocks by common-view mode.

GPS and UMTS

	GPS	UMTS
Carrier frequency	1.57542 GHz (L_1)	$\approx$ 2 GHz
Chip rate	1.023 Mcps	$\approx$ 4–5 Mcps
Access	CDMA	CDMA
Purpose	Navigation, positioning and timing	Communication, transmission of data
Critical signal parameter	Phase (time of arrival)	Amplitude (energy)
Infrastructure	Space based	Terrestrial
Ranging	One-way	Two-way (Up- and Down Link)
Receiver	As many SVs as possible	Only one base station in principle ('no' overlap design)
Propagation	Line of sight	LoS not critical, usually blocked, multipath preferred
Positioning	3D	2D
Range precision (1σ)	5 m	20–30 m (IPDL)

Source: Günter Hein (2000) *New Satellite Navigation Systems and Location Services of Terrestrial Mobile Communication—GALILEO and UMTS*. The 7th GNSS Workshop, November 30–December 2, Seoul, Korea.

Recommended Literature

Enge, P. & P. Misra (ed.s) (1999) *Proceedings of the IEEE*, **87**: 3–172

Strang, Gilbert & Kai Borre (1997) *Linear Algebra, Geodesy, and GPS*. Wellesley-Cambridge Press

Kaplan, E. D. (ed.) (1996) *Understanding GPS Principles and Applications*. Artech House

Misra, Pratap & Per Enge (2001) *Global Positioning System. Signals, Measurements, and Performance*. Ganga-Jamuna Press, in preparation

Samples of Useful Links on the Internet

GPS in General

www.navcen.uscg.mil

GPS New Signal Structure

www.ima.umn.edu/talks/workshops/8-16-18.2000

GPS Matlab files

kom.auc.dk/~borre/matlab

OEM boards

www.sirf.com

www.motorola.com/ies/GPS/products/gpsprod.html

www.topconps.com

GPS fleet management

www.thales-navigation.com

Galileo about to Move Ahead

Galileo in general

www.galileo-pgm.org

Galileo newsletters

www.genesis-office.org

EU information about Galileo

europa.eu.int/comm/energy_transport/en/gal_en.html

Source: Christian Tiberius & Peter Joosten (2001) *Galileo about to Move Ahead*. GIM International, March 2001, pp. 14–17.