Positioning Based on GPS Pseudoranges

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The ranges to GPS satellites measured by a receiver have a common bias and, therefore, are called *pseudoranges*:

\[ P_i^k = \rho_i^k + c \, dt_i + \text{noise}. \]
Observation Equation

Pseudorange on $L_1$ frequency; signal travel time $\tau^k_i = t_i - t^k$:

$$P^k_i = \rho^k_i + c \, d t_i (t) - c \, d t^k (t - \tau^k_i) + I^k_i + T^k_i - e^k_i. \quad (1)$$

The geometric distance between satellite $k$ and receiver $i$ is

$$\rho^k_i = \sqrt{(X^k - X_i)^2 + (Y^k - Y_i)^2 + (Z^k - z_i)^2}.$$ 

Neglecting satellite clock offset, ionospheric and tropospheric delays we get for $k = 1, 2, 3, 4$:

$$\begin{bmatrix} P^1_i \\ P^2_i \\ P^3_i \\ P^4_i \end{bmatrix} = \begin{bmatrix} \rho^1_i + c \, d t_i \\ \rho^2_i + c \, d t_i \\ \rho^3_i + c \, d t_i \\ \rho^4_i + c \, d t_i \end{bmatrix}.$$
The distance between satellite $k$ and receiver $i$—corrected for Earth rotation, rotation rate of the Earth is $\omega_e$—is defined by

$$
\rho_i^k = \| R_3(\omega_e \tau_i^k) \mathbf{r}^k (t - \tau_i^k)_{\text{geo}} - \mathbf{r}_i(t) \|.
$$

The matrix $R_3$ accounts for the rotation by the angle $\omega_e \tau_i^k$ while the signal is traveling:

$$
R_3(\omega_e \tau_i^k) = \begin{bmatrix}
\cos(\omega_e \tau_i^k) & \sin(\omega_e \tau_i^k) & 0 \\
-\sin(\omega_e \tau_i^k) & \cos(\omega_e \tau_i^k) & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$

The rotation is necessary when using vectors referenced to an Earth centered and Earth fixed system (ECEF).
We linearize (1) and let $0$ denote a preliminary value:

$$\begin{align*}
-X^k - X^0_i & \rho_i^k 
x_i - \frac{Y^k - Y^0_i}{(\rho_i^k)^0} y_i - \frac{Z^k - Z^0_i}{(\rho_i^k)^0} z_i + 1(c d t_i) \\
& = P^k_{i \text{obs}} - (P^k_i)^0 - e^k_i = b_i - e^k_i. \quad (2)
\end{align*}$$

The unknowns are arranged as $\mathbf{x} = (x_i, y_i, z_i, c d t_i)$ and we get

$$A \mathbf{x} = \begin{bmatrix}
-X^1 - X^0_i & \rho_1^i 
X^2 - X^0_i & \rho_2^i 
\vdots & \vdots 
X^m - X^0_i & \rho_m^i
\end{bmatrix}
\begin{bmatrix}
\frac{Y^1 - Y^0_i}{\rho_1^i} 
\frac{Y^2 - Y^0_i}{\rho_2^i} 
\vdots 
\frac{Y^m - Y^0_i}{\rho_m^i}
\end{bmatrix}
\begin{bmatrix}
\frac{Z^1 - Z^0_i}{\rho_1^i} 
\frac{Z^2 - Z^0_i}{\rho_2^i} 
\vdots 
\frac{Z^m - Z^0_i}{\rho_m^i}
\end{bmatrix}
1
x_i 
y_i 
z_i 
c d t_i
\begin{bmatrix}
1
\end{bmatrix}
= b - e. \quad (3)
$$
Introducing the unit direction vector from receiver $i$ to satellite $k$

$$(u^k_i)^0 = \left( \frac{X^k_{ECEF} - X^0_i}{\rho^k_i}, \frac{Y^k_{ECEF} - Y^0_i}{\rho^k_i}, \frac{Z^k_{ECEF} - Z^0_i}{\rho^k_i} \right)$$

we can rewrite (3) as

$$\begin{bmatrix} -(u^1_i)^0 & 1 \\ -(u^2_i)^0 & 1 \\ \vdots \\ -(u^m_i)^0 & 1 \end{bmatrix} x = b - e.$$
The least-squares solution becomes

\[
\begin{bmatrix}
\hat{x}_i \\
\hat{y}_i \\
\hat{z}_i \\
\overline{c dt_i}
\end{bmatrix} = (A^T \Sigma^{-1} A)^{-1} A^T \Sigma^{-1} b.
\]  

(4)

Often one uses \( \Sigma = \sigma^2 I \).

After a few iterations the final receiver coordinates become

\[
\hat{X}_i = X_i^0 + \hat{x}_i, \quad \hat{Y}_i = Y_i^0 + \hat{y}_i, \quad \text{and} \quad \hat{Z}_i = Z_i^0 + \hat{z}_i.
\]
A Posteriori Covariance Matrix and DOP

The a posteriori covariance matrix connected with the solution (4) is

\[
\Sigma_{\text{ECEF}} = (A^T \Sigma^{-1} A)^{-1}:
\]

\[
\Sigma_{\text{ECEF}} = \begin{bmatrix}
\sigma_X^2 & \sigma_{XY} & \sigma_{XZ} & \sigma_{X,c \, dt} \\
\sigma_{YX} & \sigma_Y^2 & \sigma_{YZ} & \sigma_{Y,c \, dt} \\
\sigma_{ZX} & \sigma_{ZY} & \sigma_Z^2 & \sigma_{Z,c \, dt} \\
\sigma_{c \, dt,X} & \sigma_{c \, dt,Y} & \sigma_{c \, dt,Z} & \sigma_{c \, dt}^2 \\
\end{bmatrix}.
\]

The matrix \( F^T \) connects Cartesian coordinate differences in the local system (at latitude \( \phi \) and longitude \( \lambda \)) and the ECEF system. The sequence \( (E, N, U) \)
assures that both the local and the ECEF systems shall be right handed:

\[ F^T = R_3(\pi)R_2(\phi - \frac{\pi}{2})R_3(\lambda - \pi) \]

\[
= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin \phi & 0 & \cos \phi \\ 0 & 1 & 0 \\ -\cos \phi & 0 & \sin \phi \end{bmatrix} \begin{bmatrix} -\cos \lambda & -\sin \lambda & 0 \\ \sin \lambda & -\cos \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} -\sin \lambda & \cos \lambda & 0 \\ -\sin \phi \cos \lambda & -\sin \phi \sin \lambda & \cos \phi \\ \cos \phi \cos \lambda & \cos \phi \sin \lambda & \sin \phi \end{bmatrix}.
\]

(6)

The covariance propagation law transforms \( \Sigma_{ECEF} \) into the covariance matrix expressed in a local system with coordinates \( (E, N, U) \). The interesting 3 by 3
submatrix $S$ of $\Sigma_{ECEF}$ is shown in (5). After the transformation $F$, the submatrix becomes

$$
\Sigma_{ENU} = \begin{bmatrix}
\sigma^2_E & \sigma_{EN} & \sigma_{EU} \\
\sigma_{NE} & \sigma^2_N & \sigma_{NU} \\
\sigma_{UE} & \sigma_{UN} & \sigma^2_U
\end{bmatrix} = F^T SF. \tag{7}
$$

In practice we meet several forms of the *dilution of precision* (abbreviated DOP):

**Geometric:**

$$
\text{GDOP} = \sqrt{\frac{\sigma^2_E + \sigma^2_N + \sigma^2_U + \sigma^2_{c\,dt}}{\sigma^2_0}} = \sqrt{\frac{\text{tr}(\Sigma_{ECEF})}{\sigma^2_0}}
$$

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Horizontal: \[
\text{HDOP} = \sqrt{\frac{\sigma_E^2 + \sigma_N^2}{\sigma_0^2}}
\]

Position: \[
\text{PDOP} = \sqrt{\frac{\sigma_E^2 + \sigma_N^2 + \sigma_U^2}{\sigma_0^2}} = \sqrt{\frac{\sigma_X^2 + \sigma_Y^2 + \sigma_Z^2}{\sigma_0^2}}
\]

Time: \[
\text{TDOP} = \sigma_c \frac{dt}{\sigma_0}
\]

Vertical: \[
\text{VDOP} = \frac{\sigma_U}{\sigma_0}.
\]

Note that all DOP values are dimensionless. They multiply the range errors to give the position errors (approximately). Furthermore we have

\[
\text{GDOP}^2 = \text{PDOP}^2 + \text{TDOP}^2 = \text{HDOP}^2 + \text{VDOP}^2 + \text{TDOP}^2.
\]
Geometric Interpretation of DOP

PDOP can be interpreted as the reciprocal value of the volume $V$ of a tetrahedron that is formed by the satellites and the receiver: $PDOP = \frac{1}{V}$. The best geometric situation exists when PDOP is minimized. With a full satellite configuration, the concept of DOP has less impact.
SA, Ionospheric and Tropospheric Delays, and Multipath

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Orbit Errors

The left plot shows the difference between precise and broadcast ephemerides. The precise ephemerides are accurate to within 5 cm. The right plot shows the length of the difference vector between precise and broadcast ephemerides as projected onto the line of sight.
Modernization of GPS

Standard Positioning Service (SPS), $\sigma$ in meters

<table>
<thead>
<tr>
<th>Error Source</th>
<th>With SA</th>
<th>Without SA</th>
<th>C/A code on L2 and/or L5</th>
<th>With AII&lt;sup&gt;a&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>SA</td>
<td>24.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>Ionosphere</td>
<td>7.0</td>
<td>7.0</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>Troposphere</td>
<td>2.0</td>
<td>0.2</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Orbit and Clock</td>
<td>2.3</td>
<td>2.3</td>
<td>2.3</td>
<td>1.25</td>
</tr>
<tr>
<td>Receiver Noise</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
</tr>
<tr>
<td>Multipath</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>User Equivalent Range Error</td>
<td>25.0</td>
<td>7.5</td>
<td>2.8</td>
<td>2.0</td>
</tr>
<tr>
<td>HDOP</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
<td>1.5</td>
</tr>
<tr>
<td>Stand-Alone, Horizontal Acc., 95%</td>
<td>75.0</td>
<td>22.5</td>
<td>8.5</td>
<td>6.0</td>
</tr>
</tbody>
</table>

Operational Control Segment improvements from 2000

Source: GPS World, September 2000, pp. 36–44

<sup>a</sup> Accuracy Improvement Initiative

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Geometry for Differential GPS

- occupy a known station
- compute range corrections
- transmit range corrections to rover
- corrections applied at rover
- improved positioning accuracy

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Differential GPS

Assume the position of receiver $i$ is known. $P^k_i$ is related to $c\, dt_i$ and can be applied as a range correction to $P^k_j$ of the rover. Combining the models at $i$ and $j$ leads to the principle of differential GPS:

\[
\begin{bmatrix}
(P^1_j \text{obs} - (P^1_j)^0) - (P^1_i \text{obs} - (P^1_i)^0) \\
(P^2_j \text{obs} - (P^2_j)^0) - (P^2_i \text{obs} - (P^2_i)^0) \\
\vdots \\
(P^m_j \text{obs} - (P^m_j)^0) - (P^m_i \text{obs} - (P^m_i)^0)
\end{bmatrix}
= \begin{bmatrix}
-(u^1_j)^0 & 1 \\
-(u^2_j)^0 & 1 \\
\vdots \\
-(u^m_j)^0 & 1
\end{bmatrix}
\begin{bmatrix}
x_j \\
y_j \\
z_j \\
c\, dt_{ij}
\end{bmatrix}.
\]

Here $c\, dt_{ij} = c\, dt_j - c\, dt_i$ is the difference of receiver clock offsets.
Wide Area DGPS (WADGSP)

WADGPS provides a powerful means for bridging the gap between single site and high-accuracy positioning in the vicinity of a correction station:

1. Monitor stations at known locations collect GPS pseudoranges from all satellites in view
2. Pseudoranges and dual-frequency ionospheric delay measurements (if available) are sent to the master station
3. Master station computes an error correction vector
4. Error correction vector is transmitted to users
5. Users apply error corrections to their measured pseudoranges and collected ephemeris data to improve navigation accuracy.

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Internet-based Global DGPS

NASA’s Global GPS Network (GGN) is operated and maintained by JPL. GGN consists of more than 60 sites in batch mode over the Internet igscb.jpl.nasa.gov. Software used:

- Real-Time Net Transfer (RTNT)
- Real-Time GIPSY (RTG).

The open Internet is a reliable choice to return GPS data for a state-space global differential system. User positions accurate to sub-meter level. gipsy.jpl.nasa.gov/igdg/system/index.html
Virtual Reference Station

Data from several reference stations are collected and processed in real-time using standard software.

Via the Internet the user asks for data from a non-existing reference station at a user specified location. The accuracy is comparable to the one obtained from a rigorous network solution.

GPS for Precise Time

GPS is the primary system for distribution of Precise Time and Time Interval (PTTI). The time scales are

- time kept by a satellite clock $t^k$
- GPS time $t_{GPS}$ defined by the Control Segment on the basis of a set of atomic standards aboard the satellites and in monitor stations
- UTC (USNO) $t_{UTC}$ the US national standard defined by the US Naval Observatory
- time kept by a user’s receiver clock $t_i$

$\delta t_{UTC} = t_{GPS} - t_{UTC}$ is currently estimated to be about 10 ns.

Synchronization of clocks by common-view mode.
## GPS and UMTS

<table>
<thead>
<tr>
<th></th>
<th>GPS</th>
<th>UMTS</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Carrier frequency</strong></td>
<td>1.57542 GHz ($L_1$)</td>
<td>$\approx$ 2 GHz</td>
</tr>
<tr>
<td><strong>Chip rate</strong></td>
<td>1.023 Mcps</td>
<td>$\approx$ 4–5 Mcps</td>
</tr>
<tr>
<td><strong>Access</strong></td>
<td>CDMA</td>
<td>CDMA</td>
</tr>
<tr>
<td><strong>Purpose</strong></td>
<td>Navigation, positioning and timing</td>
<td>Communication, transmission of data</td>
</tr>
<tr>
<td><strong>Critical signal parameter</strong></td>
<td>Phase (time of arrival)</td>
<td>Amplitude (energy)</td>
</tr>
<tr>
<td><strong>Infrastructure</strong></td>
<td>Space based</td>
<td>Terrestrial</td>
</tr>
<tr>
<td><strong>Ranging</strong></td>
<td>One-way</td>
<td>Two-way (Up- and Down Link)</td>
</tr>
<tr>
<td><strong>Receiver</strong></td>
<td>As many SVs as possible</td>
<td>Only one base station in principle ('no' overlap design)</td>
</tr>
<tr>
<td><strong>Propagation</strong></td>
<td>Line of sight</td>
<td>LoS not critical, usually blocked, multipath preferred</td>
</tr>
<tr>
<td><strong>Positioning</strong></td>
<td>3D</td>
<td>2D</td>
</tr>
<tr>
<td><strong>Range precision ($1\sigma$)</strong></td>
<td>5 m</td>
<td>20–30 m (IPDL)</td>
</tr>
</tbody>
</table>


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Recommended Literature


Samples of Useful Links on the Internet

GPS in General
www.navcen.uscg.mil

GPS New Signal Structure

GPS Matlab files
kom.auc.dk/~borre/matlab

OEM boards
www.sirf.com
www.motorola.com/ies/GPS/products/gpsprod.html
www.topconps.com

GPS fleet management
www.thales-navigation.com

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Galileo about to Move Ahead

Galileo in general

www.galileo-pgm.org

Galileo newsletters

www.genesis-office.org

EU information about Galileo

europa.eu.int/comm/energy_transport/en/gal_en.html


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